Introduction to Topology

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We’ve been looking at knot theory, which is generally seen as a branch of topology. In mathematics, topology is the study of continuous functions. So let’s investigate what it means to be continuous.

In calculus, continuous functions are ones that can’t change “too quickly” in the sense that they have limits; if the input gets close to some particular number, then the output must get close to the function at that number. Formally, the definition is:

**Definition:** The function \( f(x) \) is continuous at \( x = a \) if

a) \( f(a) = c \), and

b) For each \( \epsilon > 0 \) there is a \( \delta > 0 \) so that if \( |x - a| < \delta \) then \( |f(x) - c| < \epsilon \).

This definition is really hard to unpack, but in a nutshell it says that whenever \( x \) is close to \( a \) then \( f(x) \) is close to \( f(a) \). The epsilons and deltas are to make the idea of “close” more precise. They say that no matter how close (epsilon) I want the output to get to \( f(a) \) I can guarantee that it is that close, by getting close enough (delta) to \( a \).

Let’s say it a different way. Pick an epsilon. We want the output of \( f(x) \) to be in the interval \((c - \epsilon, c + \epsilon)\). So let’s look at the inverse image of this interval. That is, let’s look at all the possible \( x \)'s that get sent by \( f \) into this interval. The point \( a \) itself is in this inverse image, but if the function is continuous, so is a little area around \( a \).

For example, let \( f(x) = x^2 \) and \( a = 3 \). Then \( f(a) = 9 \). Pick an epsilon, say \( 1/10 \). What are all the \( x \)'s that get sent into the interval \((8.9, 9.1)\)? Of course this is the pair of intervals \((-\sqrt{9.1}, -\sqrt{8.9}) \cup (\sqrt{8.9}, \sqrt{9.1})\). Notice that this is a pair of open intervals, and that \( a = 3 \) is inside the second one. It’s not in the exact middle, but that’s not important. What is important is that we can fit an even smaller interval \((3 - \delta, 3 + \delta)\) around 3 and totally inside the inverse image.

This is always true. If \( P \) is any point in the real line and \( A \) is any open set containing \( P \), then there is a little region around \( P \) completely inside \( A \). For what do open sets look like? They look like bunches of open intervals. And if \( P \) is in the interval \((s, t)\) then all we have to do is take \( \delta \) to be the smaller of \( P - s \) and \( t - P \) and then \((P - \delta, P + \delta)\) is inside the interval \((s, t)\).

So it turns out that it’s not the epsilons and deltas that are important, but the open sets. So we’re going to worry about what happens to open sets. In fact, in the true spirit of mathematics, we’re going to generalize.

First, let’s ask how we should be able to put open sets together to get other open sets. We can glue them together and we’ll still get open sets. We can also take intersections, but here we have to be a little careful, because we can only take finite intersections.

**Example** Consider the open intervals \((-1, 1), (-1/2, 1/2), (-1/3, 1/3), \) and so forth. Their intersection is the single point 0, which is not an open set.
Definition: A topological space is a set $X$ (whose members are called points) together with a collection $\mathcal{O}$ of open sets, which are subsets of $X$ that satisfy
(a) Any union of sets in $\mathcal{O}$ is also in $\mathcal{O}$, and
(b) Any finite intersection of sets in $\mathcal{O}$ is also in $\mathcal{O}$.

Example The real line $\mathbb{R}$, together with the normal open sets are a topological space.

Example The plane $\mathbb{R}^2$, together with the normal open sets of the plane is a topological space.

Example The circle, with open sets being unions of open arcs, is a topological space.

Example All the spaces we have studied (e.g., hyperbolic space, projective plane, sphere, torus, Klein bottle, etc.) are all topological spaces. Since topology only cares about open sets and ignores specific distances, the hyperbolic place and the real plane are topologically indistinguishable. The projective plane is different, though, as is the sphere.

One thing that might be overlooked in the wording of the definition is that the word “any” might mean “empty.” What is the empty union of sets? If you take all the things that are in no sets, you get an empty set, so the empty set is an open set in any topological space. How about an empty intersection. What are the things that are in every one of no sets? Well, what’s not in this intersection? If $x$ is any point, it is in every one of the zero sets we are considering, so it’s in the intersection of them all. That is, the empty intersection is the whole set $X$.

In addition to open sets, we have the complementary concept of closed sets. In fact, it is exactly that: a set is closed if and only if its complement is open. Closed sets can be intersected arbitrarily, but only remain closed under finite unions. Can you think of an example of an infinite family of closed sets whose union is not closed?

Example In the real line, the usual open intervals and unions of them are open. The usual closed intervals are closed, as are single points. The empty set and the whole line are both open and closed. Other sets, like half-open intervals such as $[0, 3)$ are neither open nor closed.

Example For any set, we can invoke the discrete topology, where every subset is open. Also, every subset is closed. This is just a bunch of discrete points, all separated from each other—no two points are close to each other because each point has an open subset (i.e. itself!) that doesn’t include the other. On the other hand, we also have an indiscrete topology, where only the whole set and the empty set are open. All the points are now clumped together, since there are no open sets with which to separate the points.

Example There is only one topological space with one point. That point is both open and closed. There are four different topologies with two points. The indiscrete topology (where only the empty set and complete set are open), the discrete topology (where every
subset is open and closed), and two different topologies where the empty set and full set are open, as is one of the one-point subsets (one topology for each of the two points). These two topologies are the same, in the sense if we re-label the points, then we get one topology from the other.

**Example**  For a set with three points, $2^3 = 8$ subsets. Now the whole set and the empty set must both be open (and closed), but there are six others that we could choose to be open or closed. So in principle there are $2^6 = 64$ combinations. But not all are topologies. For instance, if $\{A, B\}$ and $\{B, C\}$ are open sets, then so is their intersection $\{B\}$. So how many topologies are there? How many different “kinds” of topologies are there—where relabeling the points doesn’t switch from one to another? (Answers: 29, 9 respectively)

Now for some more exotic topologies!

**Example**  For any infinite set, such as the real line, we can have the *cofinite* topology, where the open sets are exactly those sets which contain all but finite numbers of points (equivalently, the closed sets are the whole set, together with all finite sets). Points are separated from each other, because each has an open set around it which doesn’t contain the other point (name, any set the missed the other point plus any finite number of other points). But not too separated, because they don’t have such sets simultaneously—every open set around one point intersects every open set around any other point.

**Example**  A line with two origins! Consider a space assembled from the negative numbers $(-\infty, 0)$, the positive number $(0, \infty)$, and two points $P$ and $Q$. Make the open sets be: any open set of negatives, any open set of positives, and “intervals” like $(-a, 0) \cup \{P\} \cup (0, b)$ or $(-a, 0) \cup \{Q\} \cup (0, b)$, together with unions of these. These “glue” the pieces of the line together, but there are now two origins, $P$ and $Q$. They are not the same point. Each has an interval around it not containing the other, but these intervals do overlap the intervals from the other point. You could add more origins if you want!

**Example**  The plane in lexicographic order. Given two points in the plane, declare that $(a, b) < (c, d)$ if either $a < c$ or $a = c$ and $b < d$. So the points in the plane are in “dictionary order.” Then take as open sets all “intervals” between two points (not allowing either endpoint) and their unions. We allow $a, b, c, d$ to be infinite. What do open sets look like? Bands of vertical bars, including some, all, or none of each side.

Can you think of what open sets look like on some of the weirder geometries we’ve studied, such as taxicab geometry, maxi-geometry, or Paris metro geometry?