Poncelet-Steiner Theorem

We were able to get everything that compass and straightedge gives using just a compass. How about just a straightedge?

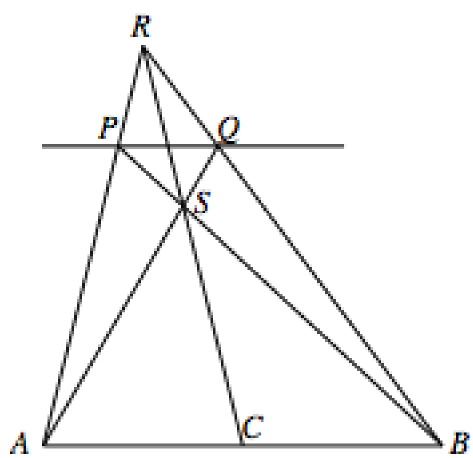
The Mohr-Mascheroni theorem that we just proved tells us that all points constructible with compass and striaghtedge can be constructed with compass alone. What if, instead, we abandon the compass?

With just straight lines, we can only solve linear equations. Thus, we can only add, subtract, multiply, and divide. And even those are hard, since it's difficult to move lengths from one place to another. But definitely no square roots. So to try to get back some normalcy, let's say someone has kindly drawn a circle for us, together with identifying its center. Now let's see how well we can do. Such constructions are called *Steiner* constuctions.

Some things don't need the circle. Watch!

Theorem 1 Given line $\overleftrightarrow{AB}$ with C the midpoint between A and B, and given point P. Then it is possible to construct the line through P parallel to $\overleftrightarrow{AB}$ using only a straightedge.

Proof: Draw a line through A and P, extended past P so some point R. Draw segments $\overline{BR}$, $\overline{CR}$, and $\overline{BP}$. Let S be the point where $\overline{CQ}$ intersects $\overline{BP}$. Then draw and extend segment $\overline{AS}$ until it intersects $\overline{BR}$ at point Q. Claim: line $\overleftrightarrow{PQ}$ is parallel to $\overline{AB}$.



Now by Ceva, $RP/PA \cdot AC/CB \cdot BQ/QR = 1$ but the middle fraction is one because C is the midpoint. So $RP/PR = RQ/QB$ so $\triangle RPQ$ is similar to $\triangle RAB$ and their bases are parallel.

Now we use our specially given circle to perform this construction even when the midpoint between A and B is not given.

Theorem 2 Given line $\overleftrightarrow{AB}$ and circle K with given center O, we can construct three points on the line that are equidistant from each other.

Proof: Draw the line from A through O. This line meets the circle in two points P and Q which are diametrically opposite. In fact, O is the midpoints between P and Q. Now pick any point X on the circle. Using Theorem 1 we can draw the line through X parallel to $\overleftrightarrow{PQ}$. Let its other point of intersection with the circle by called Y.

Now draw the diameters through X and Y, which will intersect the circle again at points X' and Y'. By construction, lines $\overleftrightarrow{XY}$, $\overleftrightarrow{PQ}$, and $\overleftrightarrow{X'Y'}$ are parallel and equally spaced. Thus they intersect $\overleftrightarrow{AB}$ in three points, of which A is the midpoint between the other two.

Thus, if we are given line $\overleftrightarrow{AB}$ and point P, we can use this construction to obtain a set of points on the line for which one is the midpoint between the other two, and then use the construction of Theorem 1 to construct the parallel through P.

Theorem 3 We can parallel translate line segments. That is, given segment $\overline{AB}$ and point P, we can translate the segment so that it remains parallel but now one endpoint is at P.

Proof: Draw the line $\overleftrightarrow{AP}$. Now draw the line parallel to $\overleftrightarrow{AB}$ through P and the line parallel to $\overleftrightarrow{AP}$ through B. These lines meet at Q, and $ABQP$ is a parallelogram so $PQ = AB$ as desired.

In the case that P lies on the same line as A and B, first pick an arbitrary point off the line to translate to, then translate again back onto the line.

Theorem 4 Given line $\overleftrightarrow{AB}$ and point P, we can Steiner-construct the line through P perpendicular to $\overleftrightarrow{AB}$.

Proof: Pick a point X on the given special circle. Draw the diameter $\overline{XZ}$ through that point, and draw the parallel to $\overleftrightarrow{AB}$ through X. This parallel again hits the circle at Y. Since $\angle XYZ$ is inscribed in a semicircle, $\overline{XY} \perp \overline{YZ}$. So now all the remains to do is draw the line through P parallel to $\overline{YZ}$, a problem we have already solved.

Theorem 5 Given points P and Q and line segment $\overline{AB}$ we can construct a segment $\overline{PR}$ whose length is equal to that of $\overline{AB}$ and which lies on $\overleftrightarrow{PQ}$ with Q and R on the same ray extending from P.

Proof: Construct a parallel to $\overline{AB}$ through P and through the center O of the special circle, which will meet the circle at X. Construct a parallel to $\overline{PQ}$ from O, meeting the circle at Y. Parallel translate $\overline{AB}$ to segment $\overline{PS}$. Finally, construct the parallel to $\overline{XY}$ through S, meeting $\overleftrightarrow{PQ}$ at R. By similar isosceles triangles $\triangle YOX$ and $\triangle SPR$ the segment $\overline{PS}$ is as promised.

Theorem 6 Given segments of lengths a, b, and s, a segment whose length is $\frac{a}{b}s$ can be constructed.

Proof: Plot arbirtary point P and two rays emanating from P. On one ray, use the previous theorem to mark point S, so that $PS = s$. On the other segment, mark points A and B so that $PA = a$ and $PB = b$. Draw line $\overleftrightarrow{SB}$. From earlier theorems, we can construct a line parallel to this through point A. This meets ray $\overrightarrow{PS}$ at T, where $AT = \frac{a}{b}s$ by similar triangles.

Theorem 7 If a is a constructible length, so is $\sqrt{a}$.

Proof: In the fixed circle, mark off a diameter, $\overline{XY}$, and let its length be d. By the previous theorem, we can mark off a length $\frac{1}{1+a}d$ starting at point X and ending at point Z on the diagonal. Then, $ZY = \frac{a}{1+a}d$. Construct the perpendicular to this diagonal through point Z. This intersects the circle at point W. By similar trianlges, $ZW = \frac{d}{a+1}\sqrt{a}$. This can be multiplied by $\frac{a+1}{d}$ to get a segment of length $\sqrt{a}$ as needed.

Alternately, on a line parallel to the diagonal, construct segment $\overline{BO}$ of length 1 and $\overline{OA}$ of length a end-to-end, as well as the perpendicular line through the common endpoint O. Then, draw the line parallal to $\overline{YW}$ through A and it will intersect the perpendicular line at C, with $OC = \sqrt{a}$, again by similar triangles.

We have now shown that we can construct all the same numbers with straightedge and fixed circle with center as we could with straightedge and compass, and of course nothing extra can be constructed in the straightedge-plus-cirlce system that we couldn't construct with a freely usable compass. So we have completed the proof of:

Theorem (Poncelet-Steiner) All constructions possible with a compass and straightedge can be completed with straightedge alone plus the use of a fixed circle whose center is known.

This theorem can be improved and variations are possible. We don't actually need the whole circle; any small arc (together with the center) will actually do—with extra work! As your homework shows, you can also get away with two circles that overlap, even if their centers are not known. You can also make due with any three non-overlapping circles. A number of other variations will also work.

An interesting question is: is this necessary? Can we get away with even less? The answer is "no". The reason is:

Theorem Using just a straightedge, it is impossible to find the center of a given circle.

Proof: In three-dimensional space, consider any two planes P_1 and P_2 and any point Q not in either plane. Define a function from P_1 to P_2 by starting with a point in P_1 and finding where the line determined by it and Q meets P_2. (There could be a line in P_1 for which the function is undefined, because the plane determined by this line and Q is parallel to P_2. This function makes more sense in projective geometry where this line would just get sent to the ideal line in P_2.) This function sends lines in P_1 to lines in P_2. In general, circles get sent to ellipses if they don't cross that bad line, hyperbolas if they do, and parabolas if they are tangent to it (remember, these are all the same in projective geometry!). Some circles, though, get sent to circles, but the center will not get sent to the center! For this circle, if there is a construction just using a straightedge to find its center, the same construction carried out in P_2 would not find the center, which is a contradiction.