Menelaus And Ceva

We investigate the Menelaus and Ceva theorems, as well as the nature of their duality.

Consider a triangle $\triangle ABC$ and select points $D$, $E$, and $F$ so that $D$ is on $\overrightarrow{BC}$, $E$ is on $\overrightarrow{AC}$, and $F$ is on $\overrightarrow{AB}$, and additionally $D$, $E$, and $F$ are collinear. There are two variations on the diagram, depending on whether the line containing $D$, $E$, and $F$ passes through $\triangle ABC$ or not.

For either of these diagrams, we have Menelaus’ Theorem:

**Theorem:** Under the conditions above, the product of the ratios

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$ 

Conversely, if the points $D$, $E$, and $F$ on the (extensions of) the sides of a triangle have their ratios multiply to $-1$ as above, the points are collinear.

The proof is quite straightforward. Drop perpendicular from $A$, $B$, and $C$ to the line through $D$, $E$, and $F$. Call the lengths of these perpendiculars $a$, $b$, and $c$ respectively. By similar triangles, $\frac{AF}{FB} = \frac{a}{b}$, $\frac{BD}{DC} = \frac{b}{c}$ and $\frac{CE}{EA} = \frac{c}{a}$ so the product of the unsigned ratios is $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = 1$. Either all three signs are negative (line is outside triangle) or exactly two are positive (line goes through triangle), so the product of the signs will be negative.

The converse can be proven by looking at point $F'$ which is where $\overrightarrow{AB}$ meets $\overrightarrow{DE}$. Replacing $F$ with $F'$ in the product of ratios must yield $-1$, and thus $\frac{AF}{FB} = \frac{AF'}{F'B}$ but we know that points that divide a line in a given ratio are unique, so $F = F'$.

Now in Menelaus we have a theorem about points and lines and collinearity, so it seems that if we move it into a projective context, there ought to be a dual version about lines and points and concurrence. In fact, the duality ought to read something like:

- Menelaus: If three points on the sides of a triangle are collinear, something interesting happens.
• Dual: If three lines through the vertices of a triangle are concurrent, something interesting happens.

This is indeed the case, and the result is **Ceva’s Theorem**:

**Theorem**  Let lines \( \overrightarrow{AD}, \overrightarrow{BE}, \) and \( \overrightarrow{CF} \) all meet (at point \( P \), say). Then if \( D, E, \) and \( F \) are the points where these lines meet the sides of the triangle,

\[
\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.
\]

The converse is also true. Note: the diagram below is the case when \( P \) is inside the triangle, but it is possible for \( P \) to be outside the triangle as well. Then two of the ratios will be negative instead of all three being positive.

While Ceva’s theorem is not hard to prove in its own right, using Menelaus gives us the result very quickly. First, use Menelaus on \( \triangle ABD \) and line \( FPC \), then again on \( \triangle ADC \) and line \( EPB \). Multiply the results and you’re done.

Oddly, it’s somewhat difficult to go the other way. To prove Menelaus from Ceva requires using Ceva six times! And like I said, there should be a joint approach using duality. The only problem is dealing with ratios of distances, because when we change points to lines, distance doesn’t make much sense.

Return to Menelaus’ Theorem, and introduce barycentric coordinates based on the original \( \triangle ABC \). Let the ratios \( AF/FB, BD/DC, \) and \( CE/EA \) be given the names \( \gamma, \alpha, \) and \( \beta \) respectively. Then, since the points \( D, E, \) and \( F \) are on the sides of the triangle and we know the ratios in which various segments are cut, we see that the barycentric coordinates for \( D, E, \) and \( F \) are \([0 : \alpha : 1], [1 : 0 : \beta] \) and \([\gamma : 1 : 0] \) respectively.

Now how do we tell if points are collinear? We check the determinant of the matrix of their barycentric coordinates. In this case, we need

\[
\begin{vmatrix} 0 & \alpha & 1 \\ 1 & 0 & \beta \\ \gamma & 1 & 0 \end{vmatrix} = 0.
\]

Computing the determinant, we find that we need \( \alpha \beta \gamma + 1 = 0 \), or \( \alpha \beta \gamma = -1 \) which is exactly Menelaus’ Theorem.

Now switch to duals. The dual of a point \([a : b : c]\) is the line with barycentric equation \( ax + by + cz = 0 \). So, for instance, the point \( D \) in the Menelaus’ version becomes the line \( \overline{AD} \) in the Ceva version, with equation \( \alpha y + z = 0 \). Now the side \( \overline{BC} \) of the triangle has
barycentric equation $x = 0$. Solving, these two lines meet at $D$ which has the barycentric coordinates $[0 : 1 : -\alpha]$. Thus, in the Ceva picture, $BD/DC = -1/\alpha$. Similarly for the other two points, so that the product of the ratios equals $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1/(\alpha \beta \gamma) = +1$. 