Introduction to Projective Geometry

Let’s change the rules of geometry to match the way we make perspective drawings. Since parallel lines appear to meet on the horizon, we’ll incorporate that idea.

Draw a picture of a large, flat desert with a pair of railroad tracks running through it. It looks something like the picture at right. When we draw a perspective picture like this, the parallel train tracks appear to actually meet at the horizon. Let’s change the rules of geometry a little so that this actually happens.

So define the projective plane to start with the ordinary plane. Each point in the plane will be an ordinary point and each line an ordinary line. To these we add a set of ideal points, one for each set of parallel ordinary lines in the plane. And we’ll take the set of ideal points and call it the ideal line.

So now every two lines meet at exactly one point. If they are intersecting ordinary lines, they meet at their ordinary intersection point. If they are parallel ordinary lines, they meet in the corresponding ideal point. If one of the lines is the ideal line, then it meets an ordinary line in the ideal point corresponding to the set of lines parallel to the ordinary line.

Similarly, every two points in the projective plane determine exactly one line. Two ordinary points determine the ordinary line as usual. An ordinary point and an ideal point determine the line taken from the set of parallel lines given by the ideal point that goes through the given ordinary point. And two ideal points determine the ideal line.

These last two paragraphs bring up two important thoughts. First, there seems now to be a duality between points and lines—every two points determine a unique line, and every two lines determine a unique point. We will see that this geometry happens to always have this property. So when we prove something for lines, there is a corresponding theorem for points and vice versa.

The second thought is that the idea that every two points determine a line is exactly our first postulate of neutral geometry. How much of the rest holds?

We run into trouble right away. The second axiom is a big problem. It says that given a point on a line and a segment between two points, we can mark off that segment along the line starting at the given point. But what if the given point on the line is the ideal point? Or what if the given segment includes an ideal point? Or two ideal points?

The other axioms hold, though, so we have circles and angles. That means we get to have trigonometry! So in this geometry even though lengths are kind of hard to study, ratios of lengths make perfect sense.

Let’s look at the situation involving just a single line. Consider the line $\overrightarrow{AB}$ with movable point $D$:
For each location of the point $D$ we will talk about the ratio $AD/DB$. In fact, we have $AD_1/D_1B = -1/4, AD_2/D_2B = 1/2, AD_3/D_3B = 2$, and $AD_4/D_4B = -4$. Note that we can have negative ratios if one of the directions is opposite the other.

Notice that if $D$ is between $A$ and $B$ then the ratio will be positive, while $A$ is between $D$ and $B$ the ratio is between 0 and $-1$. If $B$ is between $A$ and $D$ then the ratio is less than $-1$. Of course, if $D$ is at $A$, the ratio will be zero and at $B$ the ratio is undefined.

If the ratio is positive, we say that $D$ divides $AB$ in the internal ratio $AD/DB$. If the ratio is negative we’ll call it an external division. Can we get any ratio, positive or negative, that we want?

If we want a positive ratio, $r$, construct a segment $AX$ perpendicular to $\overrightarrow{AB}$ at point $A$ of length $r$. Continue this segment to $Y$ so that $XY = 1$. Draw segment $YB$ and then draw the line through $X$ parallel to this segment. It will meet $\overline{AB}$ at $D$ and by similar triangles we get $AD/DB = r$.

If the ratio needs to be negative, then instead of continuing from $A$ to $X$ to $Y$, we need to reverse direction once we get to $X$. Then $AX/XY$ will be negative.

There’s one problem. What if the ratio we desire is $-1$? Then $Y$ will be back at point $A$, and the needed line through $X$ will be parallel to the original line. So where do they intersect? The ideal point, of course! So projective geometry is just what is needed to make the idea of ratios complete.

By the way, one might think there should be two ideal points, one at each “end” of a line. I mean, if we turn around and look at the train tracks, they seem to intersect again on the opposite horizon. But it turns out this is the same ideal point. We are actually looking “all the way around” the world. That way, each pair of lines intersects in exactly one point.