An Introduction to Inversion

The operation of inversion in a circle is important and fun. We explore a few details, and apply it to hyperbolic geometry.

In Euclidean geometry you can reflect things over lines. Since lines have been replaced by circles in hyperbolic geometry, we should learn how to reflect things in circles.

Let $C$ be a circle centered at the origin with radius $r$. If $P$ is any point (other than the origin) we define its inversion $P'$ to be the point on the ray $\overrightarrow{OP}$ with (Euclidean) distance such that $OP \cdot OP' = r^2$. It is easy to find $P'$. If $P$ is inside $C$, draw a segment perpendicular to the radius through $P$. It hits $C$ at $N$. Now draw the tangent from $N$. It intersects $\overrightarrow{OP}$ at $P'$. If $P$ is outside $C$, reverse this process to find $P'$.

The figure below shows an important pair of similar triangles. Given pairs of inverse points, $A$ with $A'$ and $B$ with $B'$, the triangles $\triangle OAB$ and $\triangle OA'B'$ are similar. This is because $OA \cdot OA' = r^2 = OB \cdot OB'$ so $OA/OB = OB'/OA'$, and of course both triangles share $\angle O$. 

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Using this similarity we can prove a very important theorem.

**Theorem**: circles and lines invert to circles and lines (a line can invert to a circle and vice versa).

**Proof**: we will work by cases.

Case 1: a line through the origin inverts to itself. For points on this line invert to other points on this line, and distances are inverted (thus far points are mapped near to the origin and vice versa). So you get the entire line, except for the origin gets sent to a “point at infinity”, while that point is sent back to the origin. These extra points at infinity seem kind of fishy, but they’re harmless.

Case 2: a line not through the origin inverts to a circle through the origin and vice versa.
The figure above explains why. Let \( l \) be a line not through the origin, and let \( m \) be the line through the origin perpendicular to it. Let these intersect at point \( A \), and let \( A' \) be in inversion of this point. Now let \( B \) be any point on the line \( l \) and \( B' \) its inversion. Note that \( \angle OAB \) is a right angle, so by similarity of the triangles \( \angle OB'A' \) is also a right angle. So \( B \) varies along the line, \( B' \) must vary among points that make \( \angle OB'A' \) a right angle, and this set of points is a circle.

Case 3: a circle not through the origin inverts to another circle not through the origin. Angle chasing in the diagram below proves this result.
The same similarity picture also proves that angles are preserved under inversion. That is, if two curves meet with angle $\alpha$ between their tangents, then the inversions of these curves meet with the same angle.

This leads to an important fact: a circle that is meets the inversion circle at right angles is inverted to itself. Note the similarity to Euclidean geometry—lines perpendicular to the reflection line are reflected to themselves. This is clearly important for hyperbolic geometry. For instance, in the Poincare disk model, inversion in a geodesic keeps the unit circle invariant, so this is what reflection looks like in the hyperbolic plane.

Inversion has other uses. For instance, we can prove that if one Steiner chain beween two circles closes up, then any chain between those circles closes up, because the two circles can be inverted to be concentric, and all the circles in the chain will still be circles and all the tangency relations are preserved.