Parameterized Surfaces

Learning Goals: use the powerful technique of parameterization to start studying surfaces.

Up to now we have seen surfaces in a couple different guises:
- The graph of a function \( z = f(x, y) \)
- Implicit graphs \( g(x, y, z) = 0 \) (e.g., the sphere \( x^2 + y^2 + z^2 - 4 = 0 \))
- Things we’ve drawn but not really stated what they were (e.g. a torus).

We’d like a uniform way to discuss surfaces. Certainly graphs of functions are too restrictive, since something like a sphere isn’t one!

Just like we studied curves by parameterizing them, we can do the same with surfaces.

**Definition:** let \( D \) be a region in \( \mathbb{R}^2 \) and let \( X: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a function whose domain includes \( D \). We assume that \( X \) is one-to-one except possibly on the boundary of \( D \). The image of \( X \) is called a **parameterized surface**.

The function \( X \) will look like \((x(u, v), y(u, v), z(u, v))\). Keep in mind that the surface is a geometric object in \( \mathbb{R}^3 \) and the parameterization is an algebraic object that describes it. A surface can certainly have more than one parameterization!

**Example:** \( X_1(x, y) = (x, y, 1 - x^2 - y^2) \) defined on the unit disk in the \( xy \)-plane is one way to parameterize the upper hemisphere of the unit sphere. Another way is \( X_2(\phi, \theta) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \) with \( 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2 \).

**Example:** *any* graph of a function \( z = f(x, y) \) can be parameterized by \((x, y, f(x, y))\).

**Example:** parameterize a cone (for convenience, use a cone that opens upward, with its vertex at the origin, with height \( H \) and radius \( R \)).

**Solution:** it’s probably easiest to work in polar coordinates. So let’s start with \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \), with \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq R \). Now we just need to get the height right. At the edge, when \( r = R \), we need \( z = H \), so we set \( z = Hr/R \).

How do we draw parameterized surfaces? Probably our best bet is to use our old trick of holding one variable fixed and letting the other vary. For instance, in the cone’s parameterization \((x, y, z) = (r \cos(\theta), r \sin(\theta), Hr/R)\) if we hold \( r \) constant and let \( \theta \) vary, we see that \( x \) and \( y \) trace out a circle of radius \( r \) while \( z \) is at some fixed level depending on \( r \). We can graph these circles in three dimensions. We could also hold \( \theta \) fixed, making \( x \) and \( y \) to radiate from the origin along a straight line, while \( z \) steadily increases. The two pictures are as shown below.
You can clearly see the cone when we combine the separate pictures.

Now we notice something interesting. The function for the cone is about as differentiable as they come, but still there is a pointy end, which is not very differentiable. How did that happen? It’s related to the idea that if you have a parameterized curve it can turn a sharp angle if it first comes to a stop (velocity zero). For instance, the parameterized curve \((t^3, t^2)\) in the plane has a graph that looks like \(y = x^{2/3}\) which has a cusp at the origin. The curve can suddenly change directions because its velocity \((3t^2, 2t)\) equals zero.

So we have to figure out what’s going on with surfaces that is like this. We need to examine the tangent, apparently.

So we have a surface, and we would like to find a tangent plane. It is easy to find a tangent vector—all we need to do is find the velocity vector of one of these section curves. For instance, if we hold \(v\) fixed, we get a parameterized curve \((x(u, v_0), y(u, v_0), z(u, v_0))\) which is a function of just the variable \(u\). Its tangent vector is \(T_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)\). This vector must be tangent to our surface. So would the vector \(T_v\). But finding a couple of vectors tangent to the surface doesn’t find us the tangent plane…or does it?

Of course it does! We can find the tangent plane if we can find the normal, and we can find the normal by crossing two tangent vectors!

\[
N = T_u \times T_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.
\]

This vector is normal to our surface, and we can then find the tangent plane.

**Definition:** A surface is smooth at a point if \(N \neq 0\) at that point.

**Example:** For the parameterization \((x, y, \sqrt{1-x^2-y^2})\) of the unit sphere, find the equation of the tangent plane at the point \((2/3, 1/3, 2/3)\).
Solution: first we find the tangent vectors. \( T_u = \left(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}\right) \) which, when we plug in \((2/3, 1/3, 2/3)\) gives \((1, 0, -1)\). Similarly, \( T_v = \left(0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}\right) \) which evaluates to \((0, 1, -1/2)\).

Crossing these two gives \((1, 1/2, 1)\). So the tangent plane would be \(x + y/2 + z = 3/2\). We can confirm this is a correct equation by using our old techniques for tangent planes. If instance, using the equation \(x^2 + y^2 + z^2 - 1 = 0\), we take the gradient, and the normal vector is \((2x, 2y, 2z)\) which evaluates to \((4/3, 2/3, 4/3)\) and gives us the equation \((4/3)x + (2/3)y + (4/3)z = 2\). If we multiply this whole equation by \(\frac{3}{4}\) we get the equation found the “new” way.

What happens for the cone?

Example: for the cone parameterized by \((r \cos(\theta), r \sin(\theta), Hr/R)\) we calculate \( T_r = (\cos(\theta), \sin(\theta), H/R) \) and \( T_\theta = (-r \sin(\theta), r \cos(\theta), 0) \). Their cross product is \( N = \left(Hr \cos(\theta) / R, Hr \sin(\theta) / R, r\right)\). It is easy to see that this is an inward-pointing normal vector. But at the vertex of the cone \(r = 0\), and \(N = 0\). So the cone is not smooth at the vertex.

Caution: smoothness is a characteristic of the parameterization, not the surface itself. The parameterization \((u^3, v^3, 0)\) parameterizes the xy-plane in space, but is not smooth along the axes. That doesn’t mean the plane isn’t smooth, just that we picked a bad parameterization. Some surfaces, like the cone, cannot have a parameterization that makes them smooth, but we are not going to worry about whether we can or can’t. That’s hard work!