Introduction to Multiple Integrals

Learning Goals: we start simple, with the Riemann sum definition of an integral over a rectangle. We’ll learn how to use iterated integrals to evaluate them, and introduce Fubini’s Theorem.

**Definition:** the rectangle $R = [a, b] \times [c, d]$ is the set of all ordered pairs $(x, y)$ with $x$ in $[a, b]$ and $y$ in $[c, d]$.

Given a rectangle $R$ and a function $f$ that is reasonably nice (say, bounded and continuous except along a finite number of continuous curves in the place where it can make a [finite, since it is bounded] jump) function. Then we define the expression $\int_R f(x, y) dA$ to mean the volume under $f$ lying over the rectangle $R$. At present, we don’t know if this has any meaning or have any vague idea about how to calculate it even if it does. But please get the symbols right. There is one integral sign, you are integrating over the rectangle $R$, and you integrate with respect to $dA$. In one variable, $dx$ essentially means “a little piece of $x$,“ and if you multiply the height of the function $f(x)$ by the width of a little interval $dx$ and add, you should get the (approximate) area under a curve, and this should get better as the width of the interval gets smaller and smaller. We will do a similar thing here.

Let’s chop up both intervals into little pieces. This divides the rectangle $R$ into smaller rectangles, and we can find the height of the function over a point $(x, y)$ that lies in each small rectangle. We multiply the height of the function by the area of the small rectangle to get the volume of a small box placed under the surface. Add all these up. Such a thing is a Riemann sum for our volume.

Let $m$ be the area of the largest mini-rectangle. If in each rectangle we pick the highest point of the function, we maximize our estimate of the volume, clearly overshooting (unless the function is constant). This is called the upper Darboux sum for this division. The collection of all these upper Darboux sums for a fixed $m$ is a set of real numbers, and is bounded above (by the max of the function times the area of $R$), so by the least upper bound principal these sums have a least upper bound $U_m$. Basically, among all ways of splitting up the rectangle $R$ into pieces no bigger than $m$, none give total volumes larger than $U_m$, though some get close or are even equal to it. There is similarly a lower bound $L_m$ for all the lower Darboux sums. So for any $m$, we know that $L_m \leq \text{true volume} \leq U_m$. 
It is also fairly easy (and tedious) to prove that as \( m \) goes to zero, \( L_m \) increases and \( U_m \) decreases, and the have the same limit. We will call this limit the value of the integral above. That is, \( \int_R f \, dA \) is the limit as \( m \to 0 \) of \( L_m \) (or \( U_m \) since they produce the same limit).

**Definition:** any function that has convergence like the above is called *integrable*. The proofs we didn’t do essentially show that bounded piecewise continuous functions are integrable.

Since we can do the Riemann sum by dividing up the region in any way we want and picking the points in the little rectangles any way we want, and since all the sums involved in a Riemann sum are finite, so we can add up the terms in whatever order we want. What we are going to do is to pick a nice order. First, we will hold \( y \) steady, and let \( x \) go across all the rectangles in one little strip. Then we will pick the next \( y \), and have \( x \) run its full range again. And so forth. We can also be consistent about the way we pick the point \( (x_{ij}, y_j) \) in each little rectangle (say, always pick the lower left point) to make the Riemann sum easier to evaluate. So we can make \( x \) only depend on \( i \) and \( y \) on \( j \).

So essentially as we hold \( y \) constant at, say, \( y_j \) and evaluate \( \left( \sum_{i=1}^{n} f(x_i, y_j) \Delta x_i \right) \Delta y_j \). Since \( y \) is constant, and \( x \) only depends on \( i \), we recognize this as a Riemann sum for the integral \( \int_a^b f(x, y_j) \, dx \Delta y_j \). We can call the integral \( g(y_j) \) as \( y \) is the only thing it depends on.

Now we have one value \( g(y) \) for each \( y \) and we need to add up the sum \( \sum_{j=1}^{m} g(y_j) \Delta y_j \). This is another Riemann sum for a one-variable integral, giving \( \int_c^d g(y) \, dy \). We have proved:

**Fubini’s Theorem:** if \( f(x, y) \) is integrable, then we may evaluate \( \int_R f(x, y) \, dA \) by the *iterated* integral \( \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy \). This is also called a “double integral.”

Basically, what we are doing is slicing the shape into slices with thickness \( \Delta y \). The area of each slice can be computed by integration. Then we use the slicing idea from BC2 to compute the volume by adding up the areas of the slices times their thickness \( \Delta y \).

**Example:** let \( R \) be \([0, 2] \times [0, 3]\), and let the function \( f(x, y) = x/2 \). The volume we are computing is the wedge with base area 6 and height 1, so the volume should be \( \frac{1}{2} \cdot 6 = 3 \). We compute

\[
\int_R \frac{x}{2} \, dA = \int_0^3 \left( \int_0^2 \frac{x}{2} \, dx \right) \, dy.
\]

The inner integral gives \( \frac{x^2}{4} \). Plugging in the limits we get \( 4/4 - 0/4 = 1 \).
Then the outer integral is then \( \int_0^3 dy = 3 \).

We often simply drop the parentheses in the double integral and write \( \int_c^d \int_a^b f(x,y) dx \, dy \).

Now since that Riemann sum was a finite sum, and we can calculate it in any manner we see fit, why not hold \( x \) constant and let \( y \) vary from \( c \) to \( d \)? We would get the iterated integral \( \int_a^b f(x,y) dy \). This should give us the same answer. Fubini’s theorem says that if the function is integrable then this is indeed the case. The theorems we didn’t prove show that bounded piecewise continuous functions are integrable. So for all the reasonable functions we are dealing with we should be able to reverse the order of integration. Let’s try:

**Example:** \( \int \frac{x}{2} \, dA = \int_0^2 \left( \int_0^x \frac{1}{2} \, dy \right) \, dx = \int_0^2 \frac{xy}{2} \, dx = \int_0^2 \frac{2y}{3} \, dx = \frac{3x^2}{4} \, \bigg|_{x=0}^{x=2} = 3. \)

Note how we were careful to put “\( y = 3 \)” and so forth when noting which values go where. This is a good idea—not necessary unless it is completely ambiguous, but it does help your reader. When there are lots of variables flying around, it pays to be clear!

It is important to know that your function is integrable, or else you **cannot** evaluate integrals by iterated integration—if the integral exists at all! For instance, in the following example the function is not bounded, though it is only discontinuous at a single point:

**Example:** Let \( R = [0, 1] \times [0, 1] \). Compute \( \int_R \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dA \). If we integrate in \( y \) first, the work goes as follows:

\[
\int_R \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dA = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx = \int_0^1 \int_0^1 \frac{1 - (y/x)^2}{x^2} \, dy \, dx.
\]

Now let \( u = y/x, \, du = dy/x \), (remember, \( x \) is constant as far as the \( y \)-integral is concerned!) giving

\[
\int \int_{y=0}^{y=1} \int_{x=0}^{x=1} \frac{1}{x} \frac{1 - u^2}{(1 + u^2)^2} \, du \, dx = \int \int_{y=0}^{y=1} \frac{1}{x} \frac{1}{1 + u^2} \, du \, dx = \int \int_{y=0}^{y=1} \frac{y/x}{x} 1 + (y/x)^2 \, dx = \int \int_{y=0}^{y=1} \frac{1}{1 + x^2} \, dx.
\]

This is an easy integral; we get \( \arctan(1) - \arctan(0) = \pi/4 \). Also, you see the wisdom of saying “\( x = \)” or whichever variable, especially with the \( u \)-substitution!

What if we had done the \( x \) integral first? This time, we would pull out the \( y \) and do \( u = x/y \). Everything is the same except there is a minus sign in the integral all along, leading to \(-\pi/4\) as a final answer.

If this example function reminds you of the example we used to show that mixed partials need not be equal, good spotting! The idea of reversing mixed partials and reversing order of integration are closely related.