

Multiplicity of Eigenvalues

Learning Goals: to see the difference between algebraic and geometric multiplicity.

We have seen an example of a matrix that does *not* have a basis' worth of eigenvectors. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (note: this is *not* the Fibonacci matrix!). The characteristic polynomial of this matrix is $(1 - \lambda)^2$, so 1 is a double root of this polynomial. Finding the nullspace of $A - \lambda I$ leads us to consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, whose nullspace only contains $(1, 0)$ (and multiples of it). There is no place else to look for eigenvectors, so this matrix does not have an adequate supply!

This leads us to two definitions:

Definition: the *algebraic multiplicity* of an eigenvalue e is the power to which $(\lambda - e)$ divides the characteristic polynomial.

Definition: the *geometric multiplicity* of an eigenvalue is the number of linearly independent eigenvectors associated with it. That is, it is the dimension of the nullspace of $A - eI$.

In the example above, 1 has algebraic multiplicity two and geometric multiplicity 1. It is always the case that the algebraic multiplicity is at least as large as the geometric:

Theorem: if e is an eigenvalue of A then its algebraic multiplicity is at least as large as its geometric multiplicity.

Proof: Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r$ be all of the linearly independent eigenvectors associated to e , so that e has geometric multiplicity r . Let $\mathbf{x}_{r+1}, \dots, \mathbf{x}_n$ complete this set to a basis for \mathbf{R}^n , and let S be the matrix whose columns are \mathbf{x}_s . As usual, consider AS . Because the first few

columns of S are eigenvectors, we have $AS = \begin{bmatrix} \vdots & \vdots & & \vdots & \vdots & \vdots \\ e\mathbf{x}_1 & e\mathbf{x}_2 & \cdots & e\mathbf{x}_r & ? & \cdots & ? \\ \vdots & & & \vdots & \vdots & & \vdots \end{bmatrix}$. Now

multiply out $S^{-1}AS$. (S is invertible because its columns are a basis for \mathbf{R}^n .) We get that the first r columns of $S^{-1}AS$ are diagonal with e 's on the diagonal, but that the rest of the columns are indeterminable. Now $S^{-1}AS$ has the same characteristic polynomial as A ! This is because $\det(S^{-1}AS - \lambda I) = \det(S^{-1}AS - S^{-1}(\lambda I)S)$ (since the second product just equals λI again) $= \det(S^{-1}(A - \lambda I)S) = \det(S^{-1})\det(A - \lambda I)\det(S) = \det(A - \lambda I)$ because the determinants of S and S^{-1} cancel. So the characteristic polynomials of A and $S^{-1}AS$ are the same. But since the first few columns of $S^{-1}AS$ are a bunch of e 's on the diagonal, we see that the characteristic polynomial of $S^{-1}AS$ has a factor of at least $(\lambda - e)^r$, so the algebraic multiplicity is at least as large as the geometric.

Now we have also seen that the geometric multiplicity of each eigenvalue is at least one (there has to be *some* eigenvector). We also know by the Fundamental Theorem of Algebra that

the total number of roots of a polynomial, including multiplicities, must be the degree of the polynomial. Thus, the total algebraic degree is at most n for an $n \times n$ matrix. So there are at *most* n linearly independent eigenvectors. There are at least as many as there are different eigenvalues, but unfortunately we can't say much more than that.

When there is a basis of eigenvectors, we can diagonalize the matrix. Where there is not, we can't. We can come close, but that's another very complicated story. So the good case is when the geometric multiplicity of each eigenvalue equals its algebraic multiplicity because then there are just enough eigenvectors to make a basis.

Reading: 6.2

Problems: 6.2: 1, 2, 3, 7, 11, 12, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 35,
prove that if a 2×2 matrix with just one eigenvalue (of algebraic multiplicity two) is diagonalizable if and only if it is already a diagonal matrix