Projection Onto General Subspaces

Learning Goals: to see if we can extend the ideas of the last section to more dimensions.

So how can we accomplish projection onto more general subspaces? Let \( V \) be a subspace of \( \mathbb{R}^n \), \( W \) its orthogonal complement, and \( v_1, v_2, \ldots, v_r \) be a basis for \( V \). Put the \( v \)'s into the columns of a matrix \( A \). We want to find a way to project onto the column space of \( A \).

As before, write \( x = x_V + x_W \). Last time, we had to choose a multiple \( c \) of the basis vector \( a \) that made \( x - ca \) orthogonal to \( a \). This time we want to choose a linear combination of the \( v \)'s that makes \( x - (\text{this linear combo}) \) orthogonal to \( V \).

- A linear combination of the \( v \)'s can be written as \( A \epsilon \) for some vector \( \epsilon \)
- To check if a vector is orthogonal to \( V \), it needs only to be orthogonal to every basis vector of \( V \). In other words, it needs to be orthogonal to the columns of \( A \). In other words, it needs to be in the left nullspace of \( A \).

So the problem we need to solve is: find \( \epsilon \) so that \( x - A \epsilon \) is in the left nullspace of \( A \). Or, more simply, that \( A^T (x - A \epsilon) = 0 \).

Let’s multiply this out. We get \( A^T x - A^T A \epsilon = 0 \), or \( A^T A \epsilon = A^T x \). Since \( \epsilon \) is the only unknown here, we’re in good shape if we can find \( \epsilon \). What is the nature of its coefficient matrix \( A^T A \)? It is square, of size \( r \times r \).

**Theorem:** if the columns of \( A \) are independent, then \( A^T A \) is invertible.

**Proof:** actually, this is if and only if; if the columns of \( A \) are not independent, then \( A \) has a nontrivial null vector, which is also a nontrivial null vector of \( A^T A \), so the latter can’t be invertible. On the other hand, if \( A^T A \) is not invertible, is has a nontrivial null vector \( n \). Then \( A^T A n = 0 \). So \( n^T A^T A n = 0 \). Thus \( (A n)^T (A n) = 0 \), so \( ||A n|| = 0 \), and \( A n = 0 \). Thus the columns of \( A \) weren’t independent after all.

So the equation \( A^T A \epsilon = A^T x \) is solvable for any choice of \( x \), uniquely. So let \( \epsilon = (A^T A)^{-1} A^T x \). This is exactly what we need to multiply \( A \) by to find the projection.

**Theorem:** if the columns of \( A \) are independent, then \( x_V = A(A^T A)^{-1} A^T x \) is the projection of \( x \) onto the columns space of \( A \).

The matrix \( A(A^T A)^{-1} A^T \) is the projection matrix. Multiplication by it projects a vector into its column space. Note the similarity to the one-dimensional formula, which could very well have been written \( a(a^T a)^{-1} a^T \) because the parenthesized expression in the middle is simply a number.

Note also that all it took to solve this problem was geometry and a little clever thinking. We set up the geometric conditions for \( x_V \) to be the projection of \( x \) onto some subspace by saying it had to be a linear combination of basis vectors (thus it was in the space) and \( x - x_V \) had to be orthogonal to the subspace. These two made the computed solution unique.

**Example:** find the projection matrix onto the plane \( x + y + z = 0 \).
A basis for this plane is \((1, 0, -1)\) and \((0, 1, -1)\). Putting these as the columns of a matrix we get the following results:

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1/3 \\ 2 & -1/3 & 2/3 \end{bmatrix}, \quad (A^T A)^{-1} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}, \quad \text{and the}
\]

final projection matrix \(P = A (A^T A)^{-1} A^T = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}\). For example, if \(x = (1, 2, 3)\), \(P x = (-1, 0, 1)\). Note that the “error” \(x - Px = (2, 2, 2)\) is perpendicular to the plane \(((1, 1, 1)\) is a normal vector to the plane).

What if we had chosen a different basis, say \((1, 1, -2)\) and \((-2, 1, 1)\)? We can go through all the computations again, but we will get the same \(P\)!

**Properties of Projections**

There are some properties that projection matrices must have due to their geometric action. For instance, \(P^2 = P\). Why? Because if you project, and then project again, nothing happens the second time. The first projection is already in the subspace, so projecting it again doesn’t do anything.

This can also be seen from the formula: \(P^2 = (A (A^T A)^{-1} A^T)^2 = A (A^T A)^{-1} (A^T A) (A^T A)^{-1} A^T\), and the \((A^T A)^{-1}\) cancels with one of its inverses, leaving just the formula for \(P\).

Another property of permutation matrices is that they are symmetric: \(P^T = (A (A^T A)^{-1} A^T)^T = A^T (A^T A)^{-1} A^T\), and inverses and transposes commute, giving \(A (A^T A)^{-1} A^T\), and since the parenthesized expression is symmetric, the transpose outside it is redundant.

**Theorem:** a matrix is a projection matrix if and only if \(P = P^T = P^2\).

**Proof:** we’ve shown the only if part already. So let \(P = P^T = P^2\) and let \(V\) be the column space of \(P\). We show that \(P\) projects onto this space. Certainly for any vector \(x\), we have \(x = P x + (x - P x)\), and \(P x\) is certainly in the column space. We need to show that \(x - P x\) is orthogonal to the column space. So let \(y\) be any vector in \(V\), so that \(y = P z\) for some \(z\). Then \(y(x - P x) = y^T x - y^T P x = z^T P^T x - z^T P^T P x = z^T P x - z^T P^2 x = z^T P x - z^T P x = 0\).

**Corollary:** if \(P\) is the projection matrix onto a subspace \(V\), then \(I - P\) is the projection matrix onto its orthogonal complement.

**Proof:** \((I - P) x = x - P x\), which is exactly what’s left over when we split off the \(V\) part of \(x\) in the above proof.

Reading: 4.2
Problems: 4.2: 1, 3, 5, 6, 7, 11, 12, 14, 16, 17, 18, 19, 20, 23, 24, 29, 30