Converge or Diverge?—Ratio test

Learning goal: Use the Ratio test to compare to a geometric series.

Tell me if the following series converges or diverges: \( \frac{1}{1} + \frac{1}{13} + \frac{1}{135} + \frac{1}{1357} + \cdots \)

Well, we could try to compare it to something. Things are being multiplied together, like a geometric series, so maybe compare to a geometric? Unfortunately, \( \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \cdots \) each term of our new series is bigger than these. Maybe \( \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} + \cdots \)?

How about the series \( \sum_{n=1}^{\infty} \frac{n^3}{3^n} \)? We know geometric is dominant over powers, but is it dominant enough?

The ratio test is designed to solve just these problems. For in a true geometric sequence, the ratio of terms is always the same, \( a_{n+1} / a_n = r \), and converges if \( |r| < 1 \) and diverges otherwise. Since we’re going to deal with messy series, we’ll still keep all of our terms positive, and not worry about the possibility of negative ratios. But let’s consider the ratio of consecutive terms in each of these two example series. In the first, the ratio \( a_{n+1} / a_n = n / (2n - 1) \). Always a little larger than \( \frac{1}{2} \), but it does have a limit of \( \frac{1}{2} \). That means that eventually the ratio is always less than, say, \( \frac{3}{4} \). So from that point on, our series is less than the geometric series with \( r = \frac{3}{4} \). Since the latter converges, so does our series.

Similarly, the ratio of consecutive terms in the second example series is \( (n + 1)^3 / 3n^3 \). The limit of this is \( 1/3 \). That means the eventually the ratio is always less than, say, \( \frac{1}{2} \). From that point on, our series is less than a convergent geometric series, so also converges.

The ratio test: let \( \sum_{n=3}^{\infty} a_n \) be a sum of positive terms. Consider \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \). If this limit

- is less than one, the series converges
- is greater than one, the series diverges
- is equal to one or does not exist, no information is gained (the test is inconclusive).

The two examples above show that if the limit is less than one, then eventually our series is less than a geometric series (with a slightly larger \( r \), but still less than one), which converges. Conversely, if the limit is greater than one, our series will eventually be larger than a divergent geometric series (\( r > 1 \)). Heck, the terms won’t even go to zero, since they must continue to grow larger. Obviously, if the limit doesn’t exist we have no idea what is happening. What about if that limit is one?
Example: for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the ratio of consecutive terms approaches one, and the series converges to $\pi^2/6$. But for the series $\sum_{n=1}^{\infty} \frac{1}{n}$ the ratio of consecutive terms also approaches one, even though the series diverges. (Note that these are both $p$-types. All $p$-types have a limit of ratio of consecutive terms equal to one.)

Example: a ratio of zero: $\sum_{n=0}^{\infty} \frac{1}{n!}$ has a ratio of consecutive terms of $\frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$.

The limit of this, of course, is zero. Well—zero is less than one so this series converges (to $e$, to be precise). Note that $1! = 1$ and $0! = 1$—think about the number of ways to arrange one or zero objects in a line.

As opposed to our four-step process (show a comparison computation, tell what happens to the known series, state your comparison theorem, state your conclusion) the ratio test only requires a three-step process: 1) compute the limit of the ratio of consecutive terms, 2) (if the ratio has a limit other than one) quote the ratio test, 3) state your conclusion. The ratio test has a built-in comparison to a geometric series, so the middle two of our four steps are combined into one.

The ratio test is ideal for when factorials are in play, because of all the cancellation that usually happens. It is also great when exponential things are mixed with other things, because the other things usually go away in the limit.

Practice: (HH p. 488) 14 – 19.

Homework: Converge or Diverges, sheets 1 and 2 (1 is without ratio test, 2 is with).