Evaluating Improper Integrals

Learning goal: now that we know when an integral is improper, we will determine how to properly evaluate it.

Let’s look at \( \int_0^1 \frac{dx}{\sqrt{x}} \) which is, of course, improper at \( x = 0 \). So let’s avoid \( x = 0 \) and instead evaluate \( \int_{0.01}^1 \frac{dx}{\sqrt{x}} \). This calculation is easy: \( \int_{0.01}^1 \frac{dx}{\sqrt{x}} = 2\sqrt{1} - 2\sqrt{0.01} = 2 - 0.2 = 1.8 \).

Since that worked so well, let’s move the lower limit a little closer to zero: \( \int_{0.0001}^1 \frac{dx}{\sqrt{x}} \). This can be calculated as \( \int_{0.0001}^1 \frac{dx}{\sqrt{x}} = 2\sqrt{1} - 2\sqrt{0.0001} = 1.98 \). If we went from 0.000001 to 1, we would get 1.998, and so on. So while we can’t integrate from zero to one, because some rectangle might be too tall, we can get as close to zero as we want. Not only that, but the closer to zero we get, the less extra area we seem to be adding on. Now in fact, if we switch over to limits: \( \lim_{a \to 0^+} \int_{a}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} (2 - 2\sqrt{a}) = 2 \). It makes sense to think that the integral should equal this limit—just like any other computation in calculus we can’t really do by just plugging in zero, we do by taking a limit as the thing we would plug in goes to zero, and call this the answer. We say that \( \int_{0}^{1} \frac{dx}{\sqrt{x}} \) converges to 2.

Look at (HH 5th section 7.7 p. 378 #3) \( \int_{0}^{\infty} xe^{-x} \, dx \). Infinity is automatically a trouble spot, but otherwise this integral is perfectly nice. What do you get if instead of infinity you integrate out to 5? 10? 20? What would you predict the answer should be if we integrate out to infinity?

Now try doing the actual calculation with limits. In other words, calculate \( \lim_{b \to \infty} \int_{0}^{b} xe^{-x} \, dx \).

**RULE:** if an integral is improper at one of its endpoints, use limits to get to that endpoint. Be careful! You might have to use a one-sided limit!

Not all improper integrals have nice limits: \( \int_{1}^{\infty} \frac{dx}{x} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x} = \lim_{b \to \infty} (\ln(b) - \ln(1)) = \lim_{b \to \infty} \ln(b) \), but this has no limit. Or more precisely, this limit is infinity. As we go further to the right, even though the curve is very low to the x-axis, there is still an infinite amount of area under the curve \( y = 1/x \). (We can easily see this by taking blocks from \( x = 2^k \) to \( x = 2^{k+1} \) and putting in a single rectangle of height \( 2^{k+1} \), which has area \( \frac{1}{2} \). So we can add up more and more \( \frac{1}{2} \)'s and get an arbitrarily big total.) We say such an integral diverges.
Example: \[ \int_0^{\infty} \sin(x) \, dx \] diverges, even though nothing blows up. But there is still no nice fixed amount of area under the curve. As you go farther to the right, you get more cycles of the sine, and the total area keeps going up and down and never settles on a single value. Even though it doesn’t blow up, it still has no limit, so this integral also diverges.

Now our rule only applies to integrals which are improper at a single endpoint. What about integrals which are improper in the middle, or at both endpoints, or some combination?

Let’s take our example of \[ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}. \] This is improper at both ends. We can make our rule applicable by splitting up the integral into two pieces, say \[ \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}} + \int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}. \] Now each of these can be computed according to our rule. Now we get \[ \lim_{a \to -1} \int_{a}^{0} \frac{dx}{\sqrt{1-x^2}} + \lim_{b \to 1} \int_{0}^{b} \frac{dx}{\sqrt{1-x^2}}. \]

Please notice how there is a different variable for each limit—avoid confusion! Evaluating each piece we get \[ \lim_{a \to -1} \sin^{-1}(0) - \sin^{-1}(a) + \lim_{b \to 1} \sin^{-1}(b) - \sin^{-1}(0). \] These are all nice functions, and we get \[ 0 - (\pi/2) + \pi/2 - 0. \] Which equals \( \pi \), of course. Just like we expected all along.

Discussion: what would have happened had we broken the integral up at, say, \( \frac{1}{2} \) instead of zero?

The length better not depend on how we cut the curve into pieces, and it doesn’t.

You should now be able to complete #1, #3, and #5 on the practice sheet (as well as #2 and #6, but work on the odd ones first, then go on to the others if you finish and need to wait for the class to catch up).

Discussion: when you broke up #5 into two parts, one part converged, while the other part diverged. Does this one converge or diverge? Well, one part of it doesn’t have a meaningful answer, so the whole thing can’t have a meaningful answer. **IF ONE PART DIVERGES, THE WHOLE INTEGRAL DIVERGES!**

Let’s take a look at #4 from the practice sheet, \[ \int_{7}^{16} \frac{dx}{\sqrt{x-8}} \] which is improper in the middle. How can we make our rule apply to it? Simple—break it up where it’s improper! We will write

\[
\int_{7}^{16} \frac{dx}{\sqrt{x-8}} = \int_{7}^{8} \frac{dx}{\sqrt{x-8}} + \int_{8}^{16} \frac{dx}{\sqrt{x-8}}
\]

and now each part is again only improper at one endpoint. So we compute \( \lim_{a \to 8^{-}} \int_{a}^{8} \frac{dx}{\sqrt{x-8}} + \lim_{b \to 8^{+}} \int_{8}^{b} \frac{dx}{\sqrt{x-8}}. \) Note again how we have different letters for each limit to avoid confusion, and how we need the directions from which we are approaching 8.

So we are ready to state the full rule for evaluating improper integrals:
**RULE:** if an integral is improper at just one endpoint, you may evaluate it using a limit as you go toward that endpoint. If the integral is improper in the middle and/or at both endpoints, you need to break the interval of integration into several pieces, so that on each piece the integral is improper at just one endpoint. Remember to use different letters in each limit, and to note the direction of approach for all one-sided limits (which will be all the limits for any non-infinite endpoint of any integral). Recall that if any part diverges for any reason, the whole integral diverges and has no meaningful limit.

You are now prepared to complete the rest of the problems on the practice sheet.

**Discussion:** #7 on the worksheet diverges, but the part above the x-axis is symmetric with the part below—shouldn’t they cancel out? No! To find out why, let’s look at \( \int_{-1}^{1} \frac{dx}{x} \). Of course, we start by noting it’s improper at \( x = 0 \), so we break it up there: \( \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} \) and work out what happens to each piece. We get \( \lim_{b \to 0^+} \ln(|a|) - \ln(|b|) = \ln(1) - \ln(b) \) (you didn’t forget about the absolute values in the antiderivative of \( 1/x \) did you?). Neither of these limits exists, so according to our rule, this integral diverges.

But clearly this integral should be zero by symmetry, right? In fact, if we did things symmetrically—let our limits go to zero at the same speed—we’d get zero:

\[
\lim_{a \to 0} \left( \int_{-a}^{-1} \frac{dx}{x} + \int_{1}^{a} \frac{dx}{x} \right) = \lim_{a \to 0} \left( \ln(|a|) - \ln(|-1|) + \ln(1) - \ln(a) \right)
\]

and everything cancels. But is this fair? What if we wanted \( \int_{-1}^{1} \frac{dx}{x} \) instead? Well, since the part from \(-1\) to \(1\) is zero, we should just be left with \( \int_{1}^{2} \frac{dx}{x} \) which is \( \ln(2) \). On the other hand, if we use the same idea as above to compute the limit, we get \( \lim_{a \to 0} \left( \int_{-a}^{-1} \frac{dx}{x} + \int_{1}^{2a} \frac{dx}{x} \right) = \lim_{a \to 0} \left( \ln(|a|) - \ln(|-1|) + \ln(2) - \ln(2a) \right) \).

Inside the limit we have \( \ln(a) - 0 + \ln(2) - (\ln(2) + \ln(a)) = 0 \), not \( \ln(2) \)!

Which is right? Well, neither! If you can get different answers doing the problem different ways, it’s not a very sensible problem. The only way to resolve it is to say the more strict way of doing it is best—and if any one part diverges, the whole thing diverges.

A nice analogy is this. If you have two things that look like they might cancel each other, think of the positive one as a factory that makes cookies. Think of the negative one as a cookie monster that eats the cookies. If you have each one separately, you can’t tell what will happen—will the universe fill up with cookies, or will the cookie monster eat them at just the right pace to keep things in equilibrium, or is the cookie monster too ravenous and the factory too slow, so the monster starves? You can’t tell! But if you tie them together—put the cookie monster at the end of the production line, he will eat each cookie as it is produced, leading to no net cookies. That’s what happens when we think they should cancel out. But what if the cookie monster is a little late to work? Will he eat all the cookies by 12 (say)? If you have to know more details of
exactly what happened, it doesn’t make a lot of sense to say what should be happening without all this extra information!

Practice problems (HH 5th section 7.7 p. 378) 4, 37, 14, 16, 23, 33, 34, 39, 50
(and make sure to finish the practice sheet from class!)