**Race Track Principle**

Learning goal: a simple, almost obvious, yet very powerful consequence of the MVT

Suppose \( f(x) \) and \( g(x) \) are two functions and \( f(a) = g(a) \). Suppose further that \( f'(x) \geq g'(x) \). Then for all \( x > a \) we have that \( f(x) \geq g(x) \) (it is equally true if both inequalities are strict). Prosaically, if two horses are tied at point \( a \), and horse 1 is always going at least as fast as horse 2, then horse 1 will always be ahead of horse 2 after \( a \).

**Example:** (Hughes-Hallett section 3.10, #14) show that for all \( x > 0 \), then \( \sin(x) < x \). Well, by the racetrack principle, \( \sin(0) = 0 \), so the two functions \( f(x) = x \) and \( g(x) = \sin(x) \) are equal at \( a = 0 \). Now \( f'(x) = 1 \), while \( g'(x) = \cos(x) \leq 1 \). So \( f'(x) \geq g'(x) \) for \( x > 0 \), and racetrack tells us that \( \sin(x) \leq x \) when \( x > 0 \). To be precise, \( g'(x) < 1 \) for \( x \) just bigger than zero, so we can make the inequality strict.

**Proof of the racetrack principle:** Let \( h(x) = f(x) - g(x) \). Then \( h(0) = 0 \), and \( h'(x) = f'(x) - g'(x) \) is greater than (or equal to) zero. By the increasing function theorem, \( h(x) \) is increasing (non-decreasing), so for \( x > 0 \), \( h(x) > 0 \) (\( \geq 0 \)). So \( f \) is bigger than \( g \).

The principle can be applied in reverse (“going back in time”) for two horses that finish at the same time. Then the slower one must have been ahead.

Practice Problems: (Hughes-Hallett section 3.10) 13, 15, 19, 20, 22, (26 – 28)*