Related Rates
In the exercises 1–3, assume that both $x$ and $y$ are differentiable functions of $t$.

1. If $y = 3x$, find $\frac{dx}{dt}$ when $x = 3$ if $\frac{dy}{dt} = -2$.

   Differentiate: $dy/dt = 3dx/dt$, so if $dy/dt = -2$ then $dx/dt = -2/3$. Yes, the $x = 3$ is a red herring.

2. If $y = x^2 - 3x$, find $\frac{dx}{dt}$ when $x = 2$ if $\frac{dy}{dt} = 2$.

   Differentiate: $dy/dt = 2x \frac{dx}{dt} - 3dx/dt = (2x - 3) \frac{dx}{dt}$. Plugging in $x = 2$ and $dy/dt = 2$ gives us $dx/dt = 2$.

3. If $x^2 + y^2 = 25$, find $\frac{dx}{dt}$ when $x = 3$ and $y = -4$ if $\frac{dy}{dt} = -2$.

   Differentiate to get $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. Plugging in $x = 3$, $y = -4$ and $dy/dt = -2$ gives $6 \frac{dx}{dt} + 8 = 0$, so $dx/dt = -4/3$.

4. A boat is pulled into a dock by means of a rope. The dock is 12 feet above the deck of the boat. If the rope is being pulled in at a rate of 2 feet per second, determine the speed of the boat when 13 feet of rope are out.

   The first step should always be to draw a picture if possible. In the picture, $x$ is the horizontal distance to the boat, $y$ is the vertical distance from the boat to the dock (constant at 12 feet) and $z$ is the length of rope (currently 13 feet, changing at $-2$ feet/second).

   We can use the Pythagorean Theorem, and find $x^2 + y^2 = z^2$.

   Now if we differentiate, we find that $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$. If we plug the current values of $y$ and $z$ into the original equation, we find $x$ is currently 5. Plugging in the current value of $x$, $y$, and $z$, as well as the known derivatives $dy/dt = 0$ ($y$ isn’t changing!) and $dz/dt = -2$ feet/second, we get $2 \cdot 5 \cdot \frac{dx}{dt} + 0 = 2 \cdot 13 \cdot -2$ feet/second, so $\frac{dx}{dt} = -26/5$ feet/second.
5. A 6 foot person is walking away from a 15 foot high street light at a rate of 6 feet per second.
   a. At what rate is the length of her shadow increasing when she is 8 feet from the base of the street light?

   Let \( x \) be the distance of our walker from the lamppost, and \( y \) be the length of the shadow, as shown. We can use similar triangles to learn that \( \frac{y}{6} = \frac{x + y}{15} \). If we put the variables on separate sides of the equation, we learn that \( \frac{y}{10} = \frac{x}{15} \).
   Now differentiate: \( \frac{1}{10} \frac{dy}{dt} = \frac{1}{15} \frac{dx}{dt} \). We know that \( \frac{dx}{dt} = 6 \) feet per second, so we can conclude that \( \frac{dy}{dt} = \frac{6 \cdot 10}{15} = 4 \) feet per second.

   b. At what rate is the head of her shadow moving when she is 8 feet from the base of the street light?

   Now we basically want \( \frac{d(x + y)}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \) = 6 (given) + 4 (determined in part a) = 10 feet per second.

6. A 17 foot ladder is leaning against a wall. If the base of the ladder is pulled away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall when the base of the ladder is 5 feet from the wall? How about when the base is 10 feet from the wall? 17 feet?

   The picture here is just a triangle where the ground is a side, the wall is a side, and the ladder is the hypotenuse. Let the distance of the foot of the ladder from the wall be \( x \), and the height of the top of the ladder on the wall be \( y \). Then the Pythagorean Theorem says \( x^2 + y^2 = 17^2 \).

   Differentiate: \( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \). This becomes \( \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \). We are told that \( \frac{dx}{dt} = 2 \) feet per second. So when \( x = 5 \), we learn that \( y = \sqrt{264} \), and so \( \frac{dy}{dt} = -\frac{10}{\sqrt{264}} \) feet/second. When \( x = 10 \), \( y = \sqrt{89} \) and the speed is now \( -\frac{20}{\sqrt{89}} \) feet per second. Finally, if \( x = 17 \), then \( y = 0 \), and the speed becomes undefined.
7. A revolving beacon in a lighthouse makes one revolution every 15 seconds. The beacon is 200 feet from the nearest point (say $P$) on a straight shoreline. Find the rate at which the light beam moves along the shore at a point 400 feet down the shoreline from $P$.

The picture is shown at right; $\theta$ is the angle that the light is pointing, and $y$ is the distance from $P$ where the spot is hitting the shore. By straightforward trigonometry, $y/200 = \tan(\theta)$, or $y = 200 \tan(\theta)$. Now we want to know how fast the spot is moving, so we need $dy/dt$. Differentiating our equation gives $dy/dt = 200 \sec^2(\theta) \, d\theta/dt$. Now we plug in the known values. What is $\theta$ right now? Well, $y$ right now is 400, so $\tan(\theta) = 400/200 = 2$. We can use the identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, and get $\sec^2(\theta) = 5$. Finally, $d\theta/dt = 1 \text{ rev}/15 \text{ sec}$, or $2\pi/15 \text{ sec}$. Putting this all together, $dy/dt = 200 \text{ ft} \cdot 5 \cdot 2\pi/15 \text{ sec} = 400\pi/3 \text{ ft/second}$.

8. A stone dropped into a pond sends out a circular ripple whose radius increases at a rate of 2 feet per second. How fast is the area enclosed by the ripple increasing 3 seconds after the stone hits the water?

We know the area of a circle: $A = \pi r^2$. So differentiate: $dA/dt = 2\pi r \, dr/dt$. After three seconds, $r = 6$ feet, and $dr/dt = 2$ feet/second. So $dA/dt = 24\pi$ feet$^2$/second.

9. A tightrope is stretched 30 feet above the ground between the Em and the Saw buildings, which are 50 feet apart. A tightrope walker, walking at a constant rate of 2 feet per second from point $A$ to point $B$, is illuminated by a spotlight 70 feet above point $A$, as shown in the diagram at the right.

a. How fast is the shadow of the tightrope walker’s feet moving along the ground when she is halfway between the buildings?

Let’s add some more names to the diagram, so we can talk about things more easily. Let the position of the walker be $T$, and the point at which the spotlight hits the ground (or the Saw building) $X$. In the three diagrams below, the first illustrates part a, the second part b, and the third part c.
Now we proceed with the analysis. Let the bottom of the Em building be labeled C and call the walker’s distance from the Em building \( s = AT \) and the distance of the shadow has traveled \( r = CX \). Since \( \triangle PAT \) and \( \triangle PCX \) are similar, we know \( s/r = 7/10 \). So \( 10s = 7r \) and we differentiate to find \( 10 \frac{ds}{dt} = 7 \frac{dr}{dt} \). In the problem statement, we are given that \( \frac{ds}{dt} = 2 \) feet/second, so we solve to find \( \frac{dr}{dt} = \frac{20}{7} \) feet/second.

b. How far from point A is the tightrope walker when the shadow of her feet reaches the base of the Saw building?

A geometry problem, not calculus! In the second diagram above and with the same meanings for \( r \) and \( s \) as in part a, we see that when \( r = 50 \) then \( s = \frac{70}{10} = 35 \).

c. How fast is the shadow of the tightrope walker’s feet moving up the wall of the Saw building when she is 10 feet from point B?

Things are a little trickier this time. Let the distance \( BX \) be called \( z \). This time, the similar triangles tell us that \( z/(50 – s) = 70/s \). So we multiply out (we could take the derivative of things as they are, but who really wants to use the quotient rule??) to obtain \( z = 3500/s – 70 \). Differentiating, \( \frac{dz}{dt} = -3500s^2 \frac{ds}{dt} \). When the walker is 10 feet from \( B \), we have \( s = 40 \), and we already know \( \frac{ds}{dt} = 2 \) feet/second. So \( \frac{dz}{dt} = \frac{35}{8} \) feet per second.
10. Water is running out of a conical funnel at a rate of $3h^2$ cubic centimeters per second, where $h$ is the height of the water at time $t$. The funnel has a radius of 2 centimeters and a height of 8 centimeters (see figure). How fast is the water level dropping when the water level is 5 centimeters? [Note: the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.]

There are two ways to proceed: the easy way and the more applicable way. The easy way first:

Note that because of similar triangles, we will always have $r/h = 2/8 = 1/4$. So $r = h/4$. Since we are only concerned about $h$, we might as well get rid of all the $r$’s in the equation. So $V = \pi r^2 h/3 = \pi h^3/48$. Now differentiate: $dV/dt = \pi h^2 dh/dt/16$. We are told that $dV/dt = -3h^2$ (negative because water is running out), so we can put this in for $dV/dt$ and solve for $dh/dt$ to find $dh/dt = -48/\pi$ cm/second. Notice that this is negative—water height is decreasing.

Now in general you couldn’t so easily solve for $r$ in terms of $h$. So instead, just differentiate all the equations (don’t forget the product rule!): $dV/dt = (\pi/3)(r^2 dh/dt + 2rh dh/dt)$, and $4dr/dt = dh/dt$. Putting in $-3h^2$ for $dV/dt$ and $1/4 dh/dt$ for $dr/dt$, we get $-3h^2 = (\pi/3)(r^2 dh/dt + 2rh (1/4)dh/dt)$. Divide by $h^2$ to get $-3 = (\pi/3)(r^2/h^2 + (2/4)r/h)dh/dt$. We know $r/h = 1/4$, so put that in and solve for $dh/dt$. You better get the same answer as before!

11. Archit is flying a kite that is 100 feet above the level where he is holding the string. A wind is blowing the kite westward at a rate of 12 feet per second (i.e., the kite remains at a constant height). How fast is Archit letting out string when the kite is 200 feet away?

The picture is drawn for an eastward wind (if you assume east is right and west is left). Sorry about that!

It is not clear whether “200 feet away” means that $x = 200$ or $z = 200$. We will go with the latter. Now by Pythagoras, $100^2 + x^2 = z^2$, so differentiating gives $2x \cdot dx/dt = 2z \cdot dz/dt$. Using $z = 200$ in the Pythagorean theorem gives $x = 100\sqrt{3}$. We are also told that $dx/dt = 12$. Putting this all together tells us that $dz/dt = 2 \cdot 100\sqrt{3} \cdot 12/(2 \cdot 200) = 6\sqrt{3}$ feel per second.