Uniform Convergence and Integration

Learning goals: To show that uniform convergence is enough to guarantee limits can be interchanged with integrals. And to give an example that though sufficient, it is certainly not necessary.

Next, let’s look at the obvious theorem about integration of a sequence term-by-term. The ideal would be: if \( f_n \to f \) uniformly on \([a, b]\) and each \( f_n \) is integrable, then \( f \) is also integrable and \( \lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx \). That is, we can move the limit inside the integral.

This turns out to be completely true, and in fact we can do better (since under reasonable conditions integrability on an interval implies integrability on any subinterval as well...). The complete theorem:

**Theorem.** Let \( \alpha \) be increasing on \([a, b]\) (we’ve always had our best results with increasing \( \alpha \)!) . Let each \( f_n \in R(\alpha) \) on \([a, b]\). Define \( g_n(x) = \int_a^x f_n(t) \, d\alpha(t) \) (which we know is well-defined). Then, if \( f_n \to f \) uniformly on \([a, b]\) then

a) \( f \in R(\alpha) \) on \([a, b]\), and

b) if \( g(x) = \int_a^x f(t) \, d\alpha(t) \), then \( g_n \to g \) uniformly on \([a, b]\).

(Note: putting \( x = b \) in the above gives us the statement from the first paragraph, so this is genuinely saying a lot more!)

Note we’re using the full extent of Riemann-Stieltjes integrals, so all this applies to plain Riemann integration as well.

**Proof.** Not knowing what else to do, let’s just apply brute force. We first want to show that \( f \) is integrable, so let’s try to make its Riemann-Stieltjes sums “close to” something. Now \( f_n \), in the long run, is no more than \( \epsilon \) away from \( f \), so its integral ought not be more than \( \epsilon (\alpha(b) - \alpha(a)) \) away from the integral of \( f \) (if it exists—which we haven’t proven yet!). So try the following.

Given \( \epsilon > 0 \) choose \( N \) so large that \( |f(x) - f_N(x)| < \frac{\epsilon}{3(\alpha(b) - \alpha(a))} \) for all \( x \in [a, b] \).

Then for any given partition \( P \), the upper and lower sums satisfy \( |U(P, f - f_N, \alpha)| \leq \epsilon/3 \) and \( |L(P, f - f_N, \alpha)| \leq \epsilon/3 \). Now take a partition \( P_t \) so that for any \( P \) finer than \( P_t \) we have \( U(P, f_N, \alpha) - L(P, f_N, \alpha) < \epsilon/3 \) (we know we can do this because \( f_N \) is integrable, and thus satisfies the Riemann condition.

For these partitions, we have \( U(P, f, \alpha) - L(P, f, \alpha) \leq U(P, f - f_N, \alpha) - L(P, f - f_N, \alpha) + U(P, f_N, \alpha) - L(P, f_N, \alpha) < \epsilon \) so the upper and lower sums for \( f \) can be made as close together as we wish, meaning \( f \) satisfies the Riemann condition and is integrable. Part a) is complete.

To prove part b), choose \( \epsilon > 0 \) and choose \( N \) so that \( f_n(t) - f(t) < \epsilon/(\alpha(b) - \alpha(a)) \) for all \( n > N \) and all \( t \in [a, b] \). Then for \( x \in [a, b] \) we have \( |g_n(x) - g(x)| \leq \int_a^x |f_n(t) - f(t)| \, d\alpha(t) \leq \epsilon/(\alpha(b) - \alpha(a)) \cdot (\alpha(x) - \alpha(a)). \) since \( \alpha \) is increasing, this last is less than \( \epsilon \), proving that \( g_n \to g \) uniformly on \([a, b]\).
A simple example shows that uniform convergence is not necessary. Let $f_n(x)x^n$ on $[0, 1]$. Then the limit function is $f(x) = 0$ if $0 \leq x < 1$ but $f(1) = 1$. So even though every $f_n$ is continuous the limit is not—so the convergence can’t be uniform. But the integrals converge properly, as can easily be checked.

Oddly, the example of functions whose integrals don’t converge properly (the $g_n(x) = n^2 x(1 - x)^n$ is an example of a sequence of continuous functions whose limit function is continous, even though convergence isn’t uniform.

There is a middle ground here. The $f_n(x) = x^n$ do not converge uniformly, but neither do they get too far out of hand. At least the $f_n$ are uniformly bounded, and that bound is enough to give integrability. Basically, we are saying that because the heights of the functions aren’t too far apart, and there is pointwise convergence, then the differences in areas under the functions can’t be too much and eventually go to zero.

Definition. A sequence of functions $\{f_n\}$ boundedly converges to $f$ if it converges pointwise and is uniformly bounded.

The following theorem’s proof is very complex and detailed and will be omitted.

Theorem (Arzelà). Let $\{f_n\}$ boundedly converge to $f$ on $[a, b]$. Assume that each $f_n$ is Riemann integrable on $[a, b]$ and that the limit function $f$ is also. Then

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx = \int_a^b f(x) \, dx.$$ 

The assumption that the limit function can actually be integrated is necessary here, as the following example shows:

Example: Let $\{r_1, r_2, \ldots\}$ be a list of all the rationals between 0 and 1. Define $f_n(x)$ to be 1 if $x \in \{r_1, r_2, \ldots, r_n\}$ and 0 otherwise. Then each $f_n$ is discontinuous at a finite number of points and $\int_0^1 f_n(x) \, dx = 0$. But these functions converge boundedly to $f(x)$ which is one at every rational and zero otherwise, so is not even Riemann integrable.