11. A certain excavating machine can dig large holes. It can dig a suitable hole 10 feet deep in the first hour. Since the ground gets harder to dig as you get deeper, the machine is only able to dig 90% as many feet during each hour as the previous hour. (So it can only dig 9 additional feet during the second hour, etc.)

a. How deep is the hole after 10 hours?

\[ S_1 = 10 \]
\[ S_2 = 10 + 9 \]
\[ S_3 = 10 + 9 + 0.9(9) \]

\[ S_{10} = \frac{10(1 - 0.9^{10})}{1 - 0.9} \approx 65.132 \text{ ft} \]

b. What is the deepest hole that the machine can dig?

\[ S = \sum_{k=1}^{\infty} 10(0.9)^{k-1} \cdot \frac{10}{1 - 0.9} = 100 \]
8. For what values of \( x \) will the series \( \sum_{k=0}^{\infty} \left( \frac{x}{3} \right)^k \) have a finite sum?

\[
-1 < \frac{x}{3} < 1
\]

\[
-3 < x < 3
\]

9. Express in sigma-notation, starting at \( n = 1 \):

\[
\sum_{n=0}^{9} \frac{(2n+2)^2}{2^n} (-1)^n
\]

\[
a_n = 7 + 2(n-1)
\]

\[
= 5 + 2n
\]

10. Evaluate each of the following.

a. \( \sum_{k=1}^{40} (5 - 4k) = \frac{n}{2} \left( f + l \right) \)

\[
\frac{40}{2} \left( 5 + (-155) \right)
\]

\[
= -3020
\]

b. \( \sum_{k=26}^{80} k(k+2) = \sum_{k=26}^{80} k^2 + \sum_{k=26}^{80} k \)

\[
= \frac{80(81)(161)}{6} + \frac{(80)(81)}{2} - \left[ \frac{25(26)(51)}{6} + 2 \cdot \frac{23(24)}{2} \right]
\]

\[
= 174,185
\]
4. Find the missing terms in the harmonic sequence:
\[
\frac{2}{3}, \frac{1}{6}, \frac{2}{9}, \frac{1}{3}.
\]

5. Let \( a_n = \left( 1 - \frac{1}{n}, 1 + \frac{1}{n} \right) \), [an open interval]. Find \( a_1, a_2, \) and \( a_3 \).

\[
(0, 2), \left( \frac{1}{2}, 3 \right), \left( \frac{2}{3}, \frac{4}{3} \right)
\]

6. Clearly explain the difference between the sequence \( a_n = \{2^n\}_{n=1}^{\infty} \) and the function \( f(x) = 2^x \), where \( x \in [1, \infty) \).

The domains are different. \( \{a_n\} \) has domain \( \mathbb{N} \),

\( f \) has domain \( [1, \infty) \).

7. For a given series the sequence of partial sums \( \{S_n\}_{n=1}^{\infty} \) is given by \( S_n = \sum_{k=1}^{n} a_k \).

\[
\text{Given } S_n = \frac{5n}{2n+1}. \text{ Find } a_5.
\]

\[
\begin{align*}
a_1 &= S_1 = \frac{5}{3} \\
a_2 &= S_2 - S_1 = \frac{10}{3} - \frac{5}{3} = \frac{5}{3} \\
a_3 &= S_3 - S_2 = \frac{15}{3} - \frac{10}{3} = \frac{5}{3} \\
a_4 &= S_4 - S_3 = \frac{20}{3} - \frac{15}{3} = \frac{5}{3} \\
a_5 &= S_5 - S_4 = \frac{25}{3} - \frac{20}{3} = \frac{5}{3}
\end{align*}
\]
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You may use a TI-30 Calculator on this exam. Justify all your work.

Useful formulas: \( \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \)  \( \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4} \)

1. Given \( a_k = \begin{cases} 3, & k=1 \\ \frac{3}{2}a_{k-1} + 3, & k > 1 \end{cases} \), and \( S_n = \sum_{k=1}^{n} a_k \)
   a) State the first five terms of the sequence \( \{a_n\}_{n=1}^{5} \).
   b) State the first five terms of the sequence \( \{S_n\}_{n=1}^{5} \).

   3, 9, 21, 45, 93  
   3, 12, 34, 64, 102

2. Given a geometric sequence with \( a_3 = \sqrt{2} \) and \( a_9 = \frac{\sqrt{2}}{4} \). Solve for \( a_{12} \). Give exact answer.

   \( a_n = a_1 \cdot r^{n-1} \)

   \( r = \sqrt{2} \), \( a_1 = \frac{\sqrt{2}}{4} \)

   \( a_{12} = \frac{\sqrt{2}}{4} \cdot (\sqrt{2})^{12} = \frac{\sqrt{2} \cdot 2^6}{4} = \frac{2^5}{4} = \frac{32}{4} = \frac{16}{2} = 8 \)

   \( a_1 = \frac{16}{2} \)

3. The eighth term in an arithmetic sequence is 9 and the twenty-third term is 54. Find the sum of the first twenty-three terms.

   \( d = \frac{a_8 - a_1}{8 - 1} \)
   \( d = \frac{9 - a_1}{7} \)
   \( a_1 = 9 - 7 \cdot 3 = -18 \)

   \( a_{23} = \frac{23}{2} \left( -18 + 22 \right) \)
   \( S = \frac{23}{2} \cdot \left( -18 + 14 \right) \)
   \( = \frac{23}{2} \cdot (6) \)
   \( = 23 \cdot 3 \)
   \( = 69 \)