No Calculators allowed.
Show set-up/method clearly on all problems.

#1 (2 pts each) Suppose the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n \) converges to 5 and \( a_n > a_{n+1} > 0 \) for all \( n \geq 1 \). Indicate whether each of the following statements is T (must be true) or F (false). You may write a quick one sentence explanation, this may get you partial credit.

a. \( \square \) The sequence \( a_k \) must converge.
   \[
   \text{If the series converges, } a_k \text{ must converge to } 0.
   \]

b. \( \square \) The partial sum \( S_{101} \) could have a value of 5.01.
   \[
   S_n > 5 \text{ for all odd } n.
   \]

c. \( \square \) The series \( \sum_{n=1}^{\infty} a_n \) must converge.
   \[
   \text{By the Alternating Series Test, it is conditionally convergent.}
   \]

d. \( \square \) \( \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} \cdot a_k \right| < a_{n+1} \) for all \( n \geq 1 \).
   \[
   \text{Hence, } S - S_n \text{ can be bounded by } a_{n+1} \text{ (by AST)}
   \]

e. \( \square \) If \( f(n) = a_n \) for each \( n \geq 1 \) and \( \int_1^{\infty} f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n \) converges absolutely.
   \[
   \text{Integral test cannot be used as we don't know if } f \text{ is decreasing.}
   \]

#2 (6 pts) The series \( \sum_{k=1}^{\infty} \frac{(-1)^n}{n^n} \) converges. (This can be shown using the Alternating Series Test. But you need not show this). Find a value of \( n \), such that the partial sum \( S_n \) approximates the value of \( \sum_{k=1}^{\infty} \frac{(-1)^n}{n^n} \) with an error less than .001.

\[
|S - S_n| < a_{n+1} = \frac{1}{(n+1)^{n+1}}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>((n+1)^{n+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>256</td>
</tr>
<tr>
<td>4</td>
<td>3125</td>
</tr>
</tbody>
</table>

So, if \( n = 4 \), \( \frac{1}{(n+1)^{n+1}} = \frac{1}{3125} < \frac{1}{1000} \)

So, \( S_4 \) is within .001 of \( S \).
#3 (6 pts each) Determine whether each series converges or diverges. Explain reasoning carefully and completely.

a. \[ \sum_{n=1}^{\infty} \left[ \frac{n^5 + 3n - 1}{4n^5 - 2n^2} \right] \]

Limit Comparison: Let \( a_n = \frac{1}{n^4} \) and \( b_n = \frac{n^5 + 3n - 1}{4n^5 - 2n^2} \).

Then \( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \frac{\frac{1}{n^4}}{\frac{n^5 + 3n - 1}{4n^5 - 2n^2}} = 4 = L \)

Since \( 0 < L < \infty \) and \( \sum b_n \frac{1}{n^4} \) converges, \( \sum a_n \frac{1}{n^4} \) converges by L.C.T.

b. \[ \sum_{n=1}^{\infty} \frac{n}{3^n} \]

Ratio Test works well for this. I will use comparison test.

\[ 0 < \frac{n}{3^n} < \frac{\frac{n}{3^n}}{\frac{1}{3^n}} = \left( \frac{2}{3} \right)^n \text{ for } n > 1 \]

Since \( \sum \frac{\left( \frac{2}{3} \right)^n}{n} \) converges (geometric with \( r = \frac{2}{3} < 1 \)),

\[ \sum \frac{n}{3^n} \text{ converges by C.T.} \]

\[ \sum_{n=1}^{\infty} \left[ \frac{5^n (n!)^2}{(2n)!} \right] \]

Ratio Test: \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left[ \frac{\frac{5^{n+1} (n+1)!^2}{(2n+2)!}}{\frac{5^n (n!)^2}{(2n)!}} \right] = \lim_{n \to \infty} \frac{5 (n+1)^2}{(2n+3)(2n+1)} = \frac{5}{4} = L > 1 \)

Since \( L > 1 \), \( \sum \frac{5^n (n!)^2}{(2n)!} \) diverges by Ratio Test.
#4 (6 pts each) Determine whether each series converges absolutely, converges conditionally, or diverges. Explain reasoning carefully and completely.

a. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \]

\[ \sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \]

Since \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] converges by p-test with \( p = 2 > 1 \),

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] converges absolutely.

b. \[ \sum_{n=1}^{\infty} (-1)^n \cdot \sin \left( \frac{1}{n} \right) \]

Since \( 0 < \sin \left( \frac{1}{n} \right) < \sin(\frac{1}{\pi}) \) and \( \lim_{n \to \infty} (\sin(\frac{1}{n})) = 0 \), this series converges by A.S.T.

Look at

\[ \sum_{n=1}^{\infty} (-1)^n \left| \sin \left( \frac{1}{n} \right) \right| = \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \]

Let \( a_n = \frac{1}{n} \).

\[ \lim_{n \to \infty} \frac{a_n}{\sin \left( \frac{1}{n} \right)} = \lim_{n \to \infty} \frac{1}{\sin \left( \frac{1}{n} \right)} = 1 \]

Since \[ \sum_{n=1}^{\infty} \frac{1}{n} \] diverges, \[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \] diverges by L.C.T.

Therefore, \[ \sum_{n=1}^{\infty} (-1)^n \sin \left( \frac{1}{n} \right) \] converges conditionally.
#5(6 pts) Find an expression for the sequence of partial sums, $S_n$, for the series 
\[ \sum_{k=1}^{\infty} \cos(k \cdot \pi) \]. Use this to determine whether the series converges or diverges. If the series converges, find the value.

\[ S_n = \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cdots + \cos(n\pi) \]

\[ = -1 + 1 - 1 + 1 \cdots + (-1)^n \]

\[ = \begin{cases} 
-1 & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even.} 
\end{cases} \]

So, $S_n$ diverges.

Thus, \[ \sum_{n=1}^{\infty} \cos(k \pi) \] diverges.

#6(3 pts) Determine whether \[ \sum_{k=1}^{\infty} \frac{(2k)!}{k^k \cdot k!} \] converges or diverges. Explain all steps carefully.

Hint: Try ratio test, but be careful.