No Calculators allowed.
Show set-up/method clearly on all problems.

#1 (6pts) Find an expression for the sequence of partial sums, $S_n$, for the series
\[
\sum_{k=1}^{\infty} \left[ \frac{1}{k} - \frac{1}{k+2} \right].
\]
Use this to determine whether the series converges or diverges. If the series converges, find the value.

\[
S_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)
\]

\[
= \left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]
\]

\[
\lim_{n \to \infty} S_n = 1 + \frac{1}{2} = \frac{3}{2}
\]

So
\[
\sum_{k=1}^{\infty} \left[ \frac{1}{k} - \frac{1}{k+2} \right] = \frac{3}{2}
\]

#2 (2 pts each) Suppose the series $\sum_{n=1}^{\infty} a_n$ converges to 5 and $a_n > 0$ for all $n \geq 1$.

Indicate whether each of the following statements is T (must be true) or F (false). You may write a quick one sentence explanation, this may get you partial credit.

a. \text{\textbf{T}} The sequence $a_k$ must converge. \quad \lim_{n \to \infty} a_n = 0

b. \text{\textbf{F}} The partial sum $S_{100}$ could have a value of 5.01.

\[
\text{Since } a_n > 0, \quad S_n < S_{n+1} < \cdots \quad S = 5
\]

c. \text{\textbf{T}} The sequence of partial sums must converge to 5.

This is by definition of convergence of series.

d. \text{\textbf{F}} If the ratio test is applied to this series, then the value of $\lim_{k \to \infty} a_{k+1} / a_k$ must be a finite positive number strictly less than 1.

This limit could be equal to 1.

e. \text{\textbf{F}} If $a_k > b_k$ for all $k \geq 1$, then $\sum_{n=1}^{\infty} b_n$ must converge.

Let $b_k = -2$
#3(6 pts each) Determine whether each series converges or diverges. Explain reasoning carefully and completely.

a. \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \)

**Ratio Test:**

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n \to \infty} \frac{2}{n+1} = 0 = L
\]

Since \( L = 0 < 1 \), \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \) converges by Ratio Test.

b. \( \sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} \)

Since \( 0 < \frac{1}{n \cdot 3^n} < \frac{1}{3^n} \) and \( \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \) converges (this is a geometric series with \( r = \frac{1}{3} < 1 \)),

we know \( \sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} \) converges by C.T.

c. \( \sum_{n=1}^{\infty} \left( \frac{n^2+2n-3}{3n^3-n^2+n} \right) \)

**Limit Comparison:** Let \( a_n = \frac{1}{n} \), \( b_n = \frac{n^2+2n-3}{3n^3-n^2+n} \)

\[
\lim_{n \to \infty} \left[ \frac{a_n}{b_n} \right] = \lim_{n \to \infty} \left[ \frac{1}{n} \cdot \frac{3n^3-n^2+n}{n^2+2n-3} \right] = 3 = L
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges and \( 0 < L < \infty \), \( \sum_{n=1}^{\infty} \left( \frac{n^2+2n-3}{3n^3-n^2+n} \right) \) diverges by L.C.T.
#4 (6 pts each) Determine whether each series converges absolutely, converges conditionally, or diverges. Explain reasoning carefully and completely.

a. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)

Since \( \frac{1}{n} \to 0 \) as \( n \to \infty \), the series converges by the Alternating Series Test (A.S.T.).

Now, \( \left| \frac{1}{n} \right| = \frac{1}{n} \), the series diverges by the p-test \((p = \frac{1}{2} \leq 1)\).

So, \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \) converges conditionally.

b. \( \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n \cdot n!}{(2n)!} \)

Let's look at the ratio test: \( \left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{2^{n+1} \cdot (n+1)!}{(2n+2)!}}{\frac{2^n \cdot n!}{(2n)!}} = \frac{2^{n+1} \cdot (n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n \cdot n!} \)

So, \( \lim_{n \to \infty} \left[ \frac{a_{n+1}}{a_n} \right] = \lim_{n \to \infty} \left[ \frac{2^{n+1} \cdot (n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n \cdot n!} \right] = \lim_{n \to \infty} \left[ \frac{2(n+1)}{2(n+1)\cdot (2n+1)} \right] = 0 \leq 1 \)

So, \( \sum_{n=1}^{\infty} \frac{2^n \cdot n!}{(2n)!} \) converges by the Ratio Test.

So, \( \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n \cdot n!}{(2n)!} \) converges absolutely.
#5 (6 pts) The series $\sum_{k=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n}$ converges. (This can be shown using the Alternating Series Test. But you need not show this). Find a value of $n$, such that the partial sum $S_n$ approximates the value of $\sum_{k=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n}$ with an error less than .001.

$|S - S_n| < a_{n+1} = \frac{1}{(n+1) \cdot 2^n}$

If $n = 8$, $a_{n+1} = \frac{1}{9 \cdot 2^8} = \frac{1}{9 \cdot 512} \leq 0.001$

So, $S_8$ approximates $S$ with an error less than .001. ($S_n$ with any $n \geq 8$ will work)

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#6 (3 pts) Determine whether $\sum_{n=1}^{\infty} \frac{n! \cdot n^n}{(2n)!}$ converges or diverges. Explain reasoning carefully and completely.

**Ratio Test**

\[
\lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left[ \frac{(n+1)! \cdot (n+1)^n \cdot (2n)!}{(2n+1)! \cdot n^n} \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{(n+1) \cdot (n+1) \cdot (2n)!}{(2n+1) \cdot (2n)! \cdot n^n} \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{n+1}{2(n+1)} \cdot \frac{(n+1)^n}{n^n} \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{n+1}{2n+2} \cdot (1 + \frac{1}{n})^n \right]
\]

\[
= \frac{1}{4} \cdot e
\]

Since $L = \frac{e}{4} < 1$,

This series converges by the Ratio Test.