Clearly show ALL appropriate work for full credit. NO magically calculator leaps of faith
Calculator Allowed.

1. Consider the function \( f(x) = \sin(x^2) \).

   a. Write the first three nonzero terms and the general term of the Maclaurin series for \( f(x) \).

   \[
   \sin(x^2) = x - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots - (-1)^n \frac{x^{2n+2}}{(2n+1)!}.
   \]

   b. Use the series from part (a) to write a Maclaurin series that represents \( g(x) = \int_0^x f(t) \, dt \).

   Write the first three nonzero terms and the general term of this series.

   \[
   g(x) = \int_0^x \left( t - \frac{t^6}{3!} + \cdots \right) \, dt = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} + \cdots - (-1)^n \frac{x^{2n+3}}{(2n+3)(2n+1)!}.
   \]

2. Find an upper bound for the error made in approximating \( \ln(1.1) \) using the first six nonzero terms of the Maclaurin series for \( \ln(1 + x) \). Show/explain your analysis clearly.

\[
\left| \ln(1.1) - \left( 1 - \frac{(1.1)^2}{2} + \frac{(1.1)^3}{3} - \frac{(1.1)^4}{4} + \frac{(1.1)^5}{5} - \frac{(1.1)^6}{6} \right) \right| \leq a_{11} = \frac{(1.1)^7}{7}
\]

\[
= \frac{0.00000001}{7} \approx 4.2 \times 10^{-8}
\]
3. a. Write down the eighth degree Maclaurin polynomial for \( \cos(x) \).

\[
P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}
\]

b. What are the possible values for \( x \) if we want the error in approximating \( \cos(x) \) with the above polynomial to be less than 0.0001? Show/explain your analysis clearly.

By AST, \( \frac{x^{2n}}{2n!} \to 0 \) for \( |x| < 1 \) and \( \frac{x^{2n}}{2n!} > \frac{x^{2(n+1)}}{(2(n+1))!} \).

So,

\[
\left| \frac{x^{10}}{10!} \right| < 0.0001
\]

\[
x^{10} < \frac{1}{10000}
\]

\[
x < \frac{1}{10000}^{\frac{1}{10}}
\]

\[
\Rightarrow -1.803 < x < 1.803
\]

4. When approximating \( \frac{1}{e^2} \), how many nonzero terms of the Maclaurin series for \( f(x) = e^x \) do you have to use in order to have an error that is less than 0.001? No fair using the \( e^x \) key when solving this problem – you may use that \( 2.5 < e < 3 \).

\[
e^{-2} = 1 - 2 + \frac{2^2}{2!} - \frac{2^3}{3!} + \cdots
\]

\[
\left| e^{-2} - P_n(-2) \right| = \left| \frac{f^{n+1}(c)}{(n+1)!} \right| (0-(-1))^{n+1} = \frac{e^c}{(n+1)!} \cdot 2^{n+1} \leq \frac{e}{(n+1)!} \cdot 2^{n+1} < \frac{1}{(n+1)!} \cdot 2^{n+1}
\]

Now \( \frac{2^m}{m!} < 0.001 \) when \( m = 10 \).

So we use \( n = 9 \)

That means we have 10 terms.
5. Consider the function \( g(x) = \frac{x^2}{4 + x^2} \).

a. Write a Taylor series centered at \( x_0 = 0 \) that represents \( g(x) \). Write the first 4 terms along with the general term.

\[
\frac{x^k}{4 + x^2} - \frac{x^2}{4} \left( \frac{1}{1 + \frac{x^2}{4}} \right) = \frac{x^k}{4} \left[ 1 - \frac{x}{2} + \left( \frac{x}{2} \right)^2 - \left( \frac{x}{2} \right)^3 + \ldots \right] = \frac{x^k}{4} - \frac{x^3}{4 \cdot 2} + \frac{x^4}{4 \cdot 2^2} - \frac{x^5}{4 \cdot 2^3} + \ldots + \frac{(-1)^n}{4 \cdot 2^n} \cdot \frac{x^{n+2}}{4^n}
\]

b. Determine the interval of convergence for the series in part (a).

We need:

\[
\left( \frac{x}{\sqrt{4}} \right)^2 \leq 1
\]

\[
= \frac{x^2}{4} \leq 1
\]

\[
= \frac{-2}{2} < x < 2
\]
Question 6

Let \( f(x) = \sin(x^2) + \cos x \). The graph of \( y = |f^{(5)}(x)| \) is shown above.

(a) Write the first four nonzero terms of the Taylor series for \( \sin x \) about \( x = 0 \), and write the first four nonzero terms of the Taylor series for \( \sin(x^2) \) about \( x = 0 \).

(b) Write the first four nonzero terms of the Taylor series for \( \cos x \) about \( x = 0 \). Use this series and the series for \( \sin(x^2) \), found in part (a), to write the first four nonzero terms of the Taylor series for \( f \) about \( x = 0 \).

(c) Find the value of \( f^{(6)}(0) \).

(d) Let \( P_4(x) \) be the fourth-degree Taylor polynomial for \( f \) about \( x = 0 \). Using information from the graph of \( y = |f^{(5)}(x)| \) shown above, show that \( |P_4\left(\frac{1}{4}\right) - \frac{1}{4}\left|f^{(5)}(x)\right| < \frac{1}{3000} \).

\[
\begin{align*}
\text{(a)} \quad \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\sin(x^2) &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \\
\text{(b)} \quad \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
f(x) &= 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots \\
\text{(c)} \quad \frac{f^{(6)}(0)}{6!} \text{ is the coefficient of } x^6 \text{ in the Taylor series for } f \text{ about } x = 0. \text{ Therefore } f^{(6)}(0) = -121.
\end{align*}
\]

(d) The graph of \( y = |f^{(5)}(x)| \) indicates that \( \max_{0 \leq x \leq \frac{1}{4}} |f^{(5)}(x)| < 40 \). Therefore

\[
|P_4\left(\frac{1}{4}\right) - \frac{1}{4}| \leq \frac{\max_{0 \leq x \leq \frac{1}{4}} |f^{(5)}(x)|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.
\]