Clearly show ALL appropriate work for full credit. NO magically calculator leaps of faith. Calculator Allowed.

1. Consider the function \( f(x) = \cos(x^2) \).

   a. Write the first three nonzero terms and the general term of the Maclaurin series for \( f(x) \).

   \[
   f(x) = 1 - \frac{x^4}{2!} + \frac{x^8}{3!} - \frac{x^{12}}{4!} + \ldots + (-1)^n \frac{x^{4n}}{(2n)!}
   \]

   b. Use the series from part (a) to write a Maclaurin series that represents \( f'(x) \). Write the first three nonzero terms and the general term of this series.

   \[
   f'(x) = -\frac{4x^3}{2!} + \frac{8x^7}{3!} - \frac{12x^{11}}{4!} + \ldots + (-1)^n \frac{4n \cdot x^{4n-1}}{(2n)!}
   \]

2. Find an upper bound for the error made in approximating \( \cos(3.5) \) using the first six nonzero terms of its Maclaurin series. Show/explain your analysis clearly.

   \[
   \left| \cos(3.5) - \left( 1 - \frac{3.5^2}{2!} + \frac{3.5^4}{4!} - \frac{3.5^6}{6!} + \frac{3.5^8}{8!} - \frac{3.5^{10}}{10!} \right) \right| < a_{n+1} = \frac{3.5^{12}}{12!} , \text{ by AST since}
   \]

   \[
   \frac{3.5^n}{n!} > \frac{3.5^n}{n!} \cdot \frac{(3.5)^2}{(n+2)(n+1)} \rightarrow 0, \text{ for } n > 3 .
   \]

   So error \( \leq \frac{3.5^{12}}{12!} = .007 \)
3. \( a. \) Write down the ninth degree Maclaurin polynomial for \( \tan^{-1}(x) \).

\[
\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}
\]

\( b. \) What are the possible values for \( x \) if we want the error in approximating \( \tan^{-1}(x) \) with the above polynomial to be less than 0.01? Show/explain your analysis clearly.

\[
|\tan^{-1}(x) - P_9(x)| < a_{n+1}
\]

\[
= \left| \frac{x^{n+1}}{(n+1)!} \right| < \text{By Asymptotic} \quad \frac{x^n}{n!} \to 0 \quad \text{as} \quad \frac{x^{n+1}}{(n+1)!} < \frac{x^n}{n!}
\]

If \( |x^n| < 0.01 \)

\[
|x^n| < 0.1
\]

\[

-0.818 < x < 0.818
\]

4. When approximating \( \sqrt{e} \), how many nonzero terms of the Maclaurin series for \( f(x) = e^x \) do you have to use in order to have an error that is less than 0.0001? No fair using the \( e^x \) key when solving this problem – you may use that \( 2.5 < e < 3 \).

\[
\sqrt{e} = e^{\frac{x}{2}} \left| e^x - P_n(\frac{1}{2}) \right| < \left| \frac{f^{n+1}(c)}{(n+1)!} \left( \frac{1}{2} \right)^{n+1} \right|< \left| \frac{e^{\frac{x}{2}}}{(n+1)!} \left( \frac{1}{2} \right)^{n+1} \right|
\]

\[
\leq 0.0001
\]

\[
\text{When} \quad n+1 = 6, \quad \frac{3}{n!} \cdot \frac{1}{2^{n+1}} = 0.000065 < 0.0001
\]

So we need \( P_5(\frac{1}{2}) \).

That is, \( n=5 \) or we need \( 6 \) nonzero terms.
5. Consider the function \( g(x) = \frac{x^2}{1 + 4x^2} \).

a. Write a Taylor series centered at \( x_0 = 0 \) that represents \( g(x) \). Write the first 4 terms along with the general term.

\[
g(x) = x^2 \left( 1 - (4x^2) + (4x^2)^2 - (4x^2)^3 + \cdots \right)
= x^2 - 4x^4 + 4^2 x^6 - 4^3 x^8 + \cdots \quad (-1)^n \frac{4^n x^{2n+2}}{n!}
\]

b. Determine the interval of convergence for the series in part (a).

We need \(-1 < 4x^2 < 1\)

\[
\Rightarrow \quad x^2 < \frac{1}{4}
\]

\[
\Rightarrow \quad -\frac{1}{2} < x < \frac{1}{2}
\]
Question 6

Let \( f(x) = \sin(x^2) + \cos x \). The graph of \( y = |f^{(5)}(x)| \) is shown above.

(a) Write the first four nonzero terms of the Taylor series for \( \sin x \) about \( x = 0 \), and write the first four nonzero terms of the Taylor series for \( \sin(x^2) \) about \( x = 0 \).

(b) Write the first four nonzero terms of the Taylor series for \( \cos x \) about \( x = 0 \). Use this series and the series for \( \sin(x^2) \), found in part (a), to write the first four nonzero terms of the Taylor series for \( f \) about \( x = 0 \).

(c) Find the value of \( f^{(6)}(0) \).

(d) Let \( P_4(x) \) be the fourth-degree Taylor polynomial for \( f \) about \( x = 0 \). Using information from the graph of \( y = |f^{(5)}(x)| \) shown above, show that \( |P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)| < \frac{1}{3000} \).

\[ \begin{align*}
(a) & \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
& \quad \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \\
(b) & \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
& \quad f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots \\
(c) & \quad \frac{f^{(6)}(0)}{6!} \text{ is the coefficient of } x^6 \text{ in the Taylor series for } f \text{ about } x = 0. \text{ Therefore } f^{(6)}(0) = -121.
\end{align*} \]