Gaussian Elimination—Problems That Arise

Learning Goals: Students find out what can go wrong with elimination.

\[
x + 2y - z = 1
\]

Let’s apply elimination to the system \(2x + 4y - 3z = 1\) . After we complete eliminating \(x\)
\[
3x + 3y + z = 7
\]
\[
x + 2y - z = 1
\]
from the second and third equation we are faced with \(-z = -1\). When we go to find the
\[-3y + 4z = 4\]
next multiplier = (coefficient)/(pivot), we run into trouble because the number that is supposed to
be the pivot is zero. We can’t divide by zero, so pivots are not allowed to be zero. What can we
do to salvage this situation?

We can use one of the other two rules for altering systems to replace this zero with a
number that can be used as a pivot. Namely, we swap two rows. Here, we swap with the row
\[x + 2y - z = 1\]
below it to obtain \(-3y + 4z = 4\) . Now there is a non-zero number in the pivot position, and
\[-z = -1\]
elimination can continue normally (of course, here the elimination is finished and we may now
proceed with back substitution). So sometimes a problem may be cured by rearranging the order
of the equations.

Sometimes, though, the problem is more serious. Consider the following system, which
\[x + 2y - z = 1\]
has two different right-hand sides: \(2x + 4y - 3z = 1\) or \(1\) . After elimination of \(x\) from the
\[3x + 6y + z = 6\]
\[x + 2y - z = 1\]
second and third equations, we are left with \(-z = -1\) or \(-1\) . We’d like to apply the
\[4z = 3\]
row-switch trick, but if we look underneath the zero, all the other entries are also zero. So there
is no pivot in the second column. Not knowing what else to do, we might as well proceed to the
third column and eliminate \(z\) from the third row. This will yield \(0z = 1\) or \(0\) . In the first case, the
equations must be inconsistent, and in the second any choice of \(z\) will work so this equation was
actually redundant—it didn’t tell us anything that the other equations already didn’t tell us. In
fact, we look at the middle equation \(-z = -1\) to obtain \(z = 1\). Then we try back substitution, but
we end up with the equation \(x + 2y = 2\), and we can pick any value of \(y\) that we want and still
find an \(x\) that will work. So the system is dependent on \(y\).

So to summarize, elimination has problems if a zero occurs in a pivot position. We may
try to swap the bad row for one below it (why never above?), to get a good pivot. Any lower
row will work (not necessarily just the one directly below; you will just get a different bunch
of numbers as coefficients and right-hand sides, but the same solutions). Sometimes, though, the
breakdown of elimination is permanent, because the entire column below the trouble spot is also
zero. Then there is no fixing the system, which is either inconsistent or dependent, depending on the right-hand side.

**Side note**
This is as good a place as any to say what the third row operation (multiplying a row by a non-zero number) is good for. In theory: nothing. In practice, though, we often get into trouble when a pivot is near zero. The system might be nearly singular. This could make the system very sensitive to round-off error in the computer, for any errors that have accumulated so far are about to be divided by the tiny pivot, which will magnify them greatly. So we often “scale up” rows with a small pivot to be more compatible size-wise with the rest of the system.

Reading: 2.2  
Problems: 1 – 8 (4, 6), 11 – 16 (12, 15), 19, 21 – 23, 26, 30  
(note: these are repeated from the previous lesson)