System Solutions, Geometrically

Learning Goal: Students begin to see the interpretations of a linear system, both as the intersection of hyperplanes (the “row picture”) and as seeking the right linear combination (“column picture”)

Let’s take a closer look at one of the systems from above: \[
\begin{align*}
3a - 2b &= 5 \\
6a + b &= -10
\end{align*}
\] There are two ways we could interpret this system and its solution, \((-1, -4)\).

Row picture
The way you are familiar with looking at this system is to take each line of the system as the equation of a line in the \(ab\)-plane. Each line contains all the solutions to one of the two equations. So where the lines intersect must contain the simultaneous solution to the equations. This is shown in the diagram at right.

Column picture
There is another way to picture the system and its solution. We could rewrite the equation as \[
\begin{bmatrix} 3 \\ 6 \end{bmatrix}a + \begin{bmatrix} -2 \\ 1 \end{bmatrix}b = \begin{bmatrix} 5 \\ -10 \end{bmatrix}. \]
In other words, we seek a linear combination of two given vectors that adds up to a third given vector.

We see from the picture that since \((-2, 1)\) and \((3, 6)\) point in different directions, we can “fill up” the whole plane with linear combinations of them. We sketch lines parallel to these vectors through \((5, -10)\) and then find out what multiples of \((-2, 1)\) and \((3, 6)\) hit these lines. We find that \(-1(3, 6) + -4(-2, 1)\) adds up to \((5, -10)\), so this is the solution we seek. Notice, of course, that we have the same \(a\) and \(b\) as before.

In general
Every system has these dual points of view. We can look at the system by rows, and each row is an equation with some solution set. We seek the intersection of these solution sets. We can also look at the system by columns. Each column is a vector, and we try to find a linear combination of these vectors that ends up at our intended right-hand side.