Conics in the Projective Plane

Vincent J. Matsko

Draft

January–February 2008

Disclaimer: These notes are not meant to be a complete introduction, but rather remarks to myself about notations and ideas from [Salmon]. Read at your own risk!

1 The Projective Plane

There are various models for the projective plane, and various reasons for choosing a particular one.

Conceptually the simplest is to define "points" as lines through the origin of three-dimensional space and "lines" as planes through the origin of the same space. Thus, two distinct points determine a line (the plane containing the two lines), and two distinct lines intersect in a point (the line which is the intersection of the two planes through the origin).

Usually, though, this model is confined to (the surface of) a sphere, so that "points" are pairs of antipodal points on the sphere and "lines" are the great circles on the sphere.

A hemisphere model may also be constructed by identifying antipodal points on a sphere, so that the projective plane is an open hemisphere together with a half-open semi-circle on its boundary.

The model we will consider here is the Euclidean plane together with a "line at infinity." The reason is simple: we want to study conic sections, and in particular, we want to draw plenty of neat graphics to illustrate various interesting results in [Salmon]. In creating computer graphics and writing line-drawing algorithms, questions such as "When does a conic have a vertical tangent?" are important, since naive algorithms will attempt to divide by 0 in such cases, with predictable results. And most computer graphics tools use a Cartesian coordinate system (modeling Euclidean geometry), so that using a model that may easily be interpreted in Euclidean terms is desirable.

Fortunately, it turns out that many results about tangents may be easily expressed in projective terms, then interpreted in a Euclidean context.

To make this model, we need some "new" points on the "line at infinity." So we call a point (x_1, x_2) in the Euclidean plane $(x_1, x_2, 1)$ in the projective plane, and we say that two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, with x_3 and y_3 not 0, are **equivalent**, and write $x \equiv y$, if there is $k \in \mathbb{R}$ with $x = ky$.

What of a point with $x_3 = 0$? Let's consider for a moment all points satisfying the equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0.$$

Points with $x_3 = 1$ will correspond to a set of points on a Euclidean line. However, it is clear that the point $(\lambda_2, -\lambda_1, 0)$ is also on this line. Such a point is on the **line at infinity**, and is on every line with equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3' x_3 = 0.$$

Said in another way, *every* line with "Euclidean" slope $-\lambda_1/\lambda_2$ passes through the point $(\lambda_2, -\lambda_1, 0)$, so that there are no parallel lines in projective geometry: two lines parallel in the Euclidean sense intersect in a point on the line at infinity.

Thus, we have $x_3 = 0$ as an equation for the line at infinity. There is no point with coordinates $(0, 0, 0)$. For more information, consult [Gibson, Chapter 9].

2 Linear Algebra

2.1 Coordinates

In working with homogeneous coordinates, the use of vectors and various ideas from linear algebra is especially fruitful. But several caveats are in order.

For example, suppose points

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

are given. What sense does it make to write "$2x$"? Recall that $2x$ is, as a vector, just a different way to write x. Thus, we say that

$$x \equiv 2x,$$

or more generally,

$$x \equiv kx, \quad x \neq \mathbf{0}, \quad k \neq 0. \tag{1}$$

It is important to note that we are using the same symbol, "x," for both a point and its coordinates. It is unlikely that any confusion is possible.

Now suppose λ is the line containing x and y, with

$$\lambda = (\lambda_1, \lambda_2, \lambda_3).$$

Then, considered as purely linear algebraic objects,

$$\lambda x = \lambda y = (0).$$

But, as is typical, we identify 1×1 matrices with the real numbers, and instead write

$$\lambda x = \lambda y = 0.$$

Even though computation of λx "looks like" taking a dot product, we don't have a clear notion of angles as yet in projective space, so it doesn't really help to think of $\lambda x = 0$ as meaning "λ is perpendicular to x," but rather, x lies on λ (where, again, we safely use the same symbol, "λ," for both the line and its coordinates).

We also write

$$\lambda \equiv k\lambda, \quad \lambda \neq \mathbf{0}, \quad k \neq 0 \tag{2}$$

to indicate that $k\lambda$ is also a representation for λ.

Note that points are represented as "column vectors" and lines are represented by "row vectors" so that expressions like λx make sense as algebraic objects.

Further, though we use vector notation, the points in projective space certainly do *not* form a vector space in any meaningful say. For one, there is no zero. Moreover, we may write

$$x + x \equiv x,$$

but we could not, as a result, conclude that $x \equiv \mathbf{0}$, as would be expected in a vector space.

2.2 Joins and Intersections

Let points

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

be given. To find coordinates for the line λ joining x and y, we must essentially solve the linear system

$$\lambda x = 0, \quad \lambda y = 0.$$

As this is a system of two equation in the three variables λ_1, λ_2, and λ_3, there are an infinite number of solutions. One is given by

$$(x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1),$$

as can be easily verified. But this is the linear algebraic object we know as the **cross product**, and recall that it does indeed give us a vector (λ) "perpendicular" to two other vectors (x and y). Thus it is very useful to employ linear algebraic ideas as long as we are *very* careful about how we interpret them.

So if λ is the line joining x and y, we may write

$$\lambda \equiv x \times y.$$

Again, the cross product is used more as an algebraic convenience, and the use of "$\equiv$" is especially important here, since λ could be any nonzero multiple of $x \times y$. It is also important to note that when x and y are column vectors, we define $x \times y$ to be a row vector, so that the expressions

$$(x \times y)x = 0, \quad (x \times y)y = 0$$

make sense.

We can employ the same trick when we know that x is the intersection of the two lines λ and μ, since then we must solve the system

$$\lambda x = 0, \quad \mu x = 0,$$

giving

$$x \equiv \lambda \times \mu.$$

Again, note the use of "$\equiv$," and also note that for two row vectors, their cross product is defined as a column vector, so that expressions such as

$$\lambda(\lambda \times \mu) = 0, \quad \mu(\lambda \times \mu) = 0$$

make sense.

We may use other properties of the cross product as they apply; for example,

$$x \times y = -y \times x.$$

But note here the use of "$=$," since these expressions are in fact equal and not merely multiples of one another.

2.3 Lines and Pencils

Let x and y be two distinct points on the line λ. Since $\lambda x = \lambda y = 0$, it is evident that

$$\lambda(px + qy) = 0$$

for any p and q, not both zero. Thus, for p and q not both 0, $px + qy$ lies on the line joining x and y. Letting $p = 1$, we see that $x + qy$, as q ranges over $\mathbb{R}$, parameterizes all points on λ except for y.

In particular, since x and $x + y$ are both on λ, we see that

$$\lambda \equiv x \times (x + y).$$

But again, a comment is in order. Since as algebraic objects,

$$x \times (x + y) = x \times x + x \times y = \mathbf{0} + x \times y = x \times y,$$

and since there is no $\mathbf{0}$ in projective space, some sense must be made of expressions like "$x \times (x + y)$."

Of course the expression "$x \times x$" makes little sense as there is not a unique line joining x and x. But there is a way to make sense of

$$x \times (x + y) = x \times y.$$

If x and y are distinct, then so are x and $x + y$. But $x + y$ is on λ. Thus, the line joining x and $x + y$ is in fact the same line as that joining x and y.

But this remark merely states that

$$x \times (x + y) \equiv x \times y.$$

The two expressions are in fact equal as linear algebraic objects, therefore justifying the stronger statement.

Of course parallel statements may be made concerning two distinct lines λ and μ intersecting at x. Then $p\lambda + q\mu$, with p and q not both 0, is a generic line passing through x. Statements like $\lambda \times (\lambda + \mu) = \lambda \times \mu$ also make sense because x is intersection of the distinct lines λ, μ, and $\lambda + \mu$.

It pays to notice the principle of duality at work here. Notice how statements made about two distinct points on a line have exact parallels to statements made about two distinct lines intersecting in a point. This is one great advantage of considering conics in projective space as well as Euclidean space.

2.4 Lines and Points in General Position

On occasion, we will need to consider three lines, not concurrent, and their three points of intersection (or dually, three points, not collinear, and the three distinct lines they form). So consider lines λ_1, λ_2, and λ_3, and their intersection points x_1 (the intersection of λ_2 and λ_3), x_2 (λ_1 and λ_3), x_3 (λ_1 and λ_2). As a result of previous discussions, it is clear that we have:

$$x_1 \equiv \lambda_2 \times \lambda_3, \quad x_2 \equiv \lambda_3 \times \lambda_1, \quad x_3 \equiv \lambda_1 \times \lambda_2$$

and

$$\lambda_1 \equiv x_2 \times x_3, \quad \lambda_2 \equiv x_3 \times x_1, \quad \lambda_3 \equiv x_1 \times x_2.$$

Now consider that point and line coordinates are equivalent up to nonzero multiples. It is worthwhile to ask: can all of the equivalences ($\equiv$) in the above relationship actually be equalities ($=$)?

We are happy to answer in the affirmative. The trick is to scale the λ_i so that

$$\lambda_i x_i = 1, \quad i = 1, 2, 3.$$

This is possible since x_i does not lie on λ_i, and hence $\lambda_i x_i$ can never be 0.

More formally, suppose that three noncollinear points x_1, x_2, and x_3 be given. Define λ_1, λ_2 and λ_3 so that

$$\lambda_1 = x_2 \times x_3, \quad \lambda_2 = x_3 \times x_1, \quad \lambda_3 = x_1 \times x_2$$

and

$$\lambda_1 x_1 = 1, \quad \lambda_2 x_2 = 1, \quad \lambda_3 x_3 = 1.$$

Then

$$\begin{aligned} x_2 \times x_3 &= (\lambda_3 \times \lambda_1) \times x_3 \\ &= (\lambda_3 x_3)\lambda_1 - (\lambda_1 x_3)\lambda_3 \\ &= \lambda_1 \end{aligned}$$

since $\lambda_3 x_3 = 1$ and $\lambda_1 x_3 = 0$ since x_1 is on λ_3. The second expression is a property of the cross product often written in the form

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}.$$

The symmetry of the definitions implies that $x_3 \times x_1 = \lambda_2$ and $x_1 \times x_2 = \lambda_3$ as well.

3 Euclidean Embedding

3.1 Introduction

Once this coordinatization of projective space is introduced, is it possible to make any sense of Euclidean geometry within this framework? More specifically, is it possible that certain Euclidean problems are actually *easier* to solve in a projective context?

In order to find out, it is necessary to be able to discuss Euclidean objects in the language of projective geometry. Let's see how it goes.

To make our work easier, let us mean by a **Euclidean line** a line other than the line at infinity, and by a **Euclidean point** a point not on the line at infinity. Thus, for a Euclidean line $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, λ_1 and λ_2 are not both 0, and for a Euclidean point $x = (x_1, x_2, x_3)^t$, $x_3 \neq 0$.

3.2 Parallel Lines

Of course there are no parallel lines in projective geometry, as every two distinct lines intersect in exactly one point. But two lines which are parallel in a Euclidean sense intersect, in a projective context, at a point at infinity.

So given two lines λ and μ, recall that their intersection is given by $\lambda \times \mu$. This point must lie on the line at infinity, λ_∞, motivating:

Definition: *Two Euclidean lines λ and μ are said to be* **parallel** *if*

$$\lambda_\infty(\lambda \times \mu) = 0. \tag{3}$$

Of course

$$\lambda_\infty(\lambda_\infty \times \mu) = 0$$

for all lines μ as $\lambda_\infty \times \mu$ must lie on both λ_∞ and μ. But the line at infinity is the line "added" to Euclidean geometry, so we consider lines other than λ_∞.

3.3 Perpendicular Lines

Let's begin by considering what it means for two lines to be perpendicular in Euclidean geometry. There are different ways to consider this, but as we are dealing with equations of lines, it is useful to say that lines are perpendicular if their slopes are opposite reciprocals, more or less. Thus, generically, perpendicular lines have equations

$$Ax + By + C = 0, \quad kBx - kAy + kD = 0, \quad k \in \mathbb{R}.$$

In a projective context, these equations become

$$Ax_1 + Bx_2 + Cx_3 = 0, \quad kBx_1 - kAx_2 + kDx_3 = 0.$$

Thus,

$$A(kB) + B(-kA) + C(kD) = C(kD).$$

Now let's frame this in a projective context. Of course we must distinguish the line at infinity, since it is the only "non-Euclidean" line. There are a few ways to do this. The first is to define

$$\mathbf{n} := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and rephrase the above statement as: for two lines λ and μ, we have

$$\lambda\mu^t = (\lambda\mathbf{n}^t)(\mu\mathbf{n}^t). \tag{4}$$

Since $\lambda_\infty \equiv \mathbf{n}$, we may also say that

$$(\lambda\mu^t)(\lambda_\infty\lambda_\infty^t) = (\lambda\lambda_\infty^t)(\mu\lambda_\infty^t). \tag{5}$$

It is often useful to use (4) instead of (5).

(4) may also be written as

$$((\lambda \times \mathbf{n}) \times \mu^t)\mathbf{n}^t = 0, \tag{6}$$

and (5) may be rewritten as

$$((\lambda \times \lambda_\infty) \times \mu^t)\lambda_\infty^t = 0. \tag{7}$$

3.4 Perpendicular Lines Through a Point

Now suppose we wish to find the line through a point x perpendicular to the line $\lambda = (A, B, C)$. Assume that A and B are not both 0, so that λ is a Euclidean line. Now it is clear that $(-B, A, 0)^t$ is the point at infinity on λ. Continuing, it becomes clear that $(A, B, 0)^t$ is the point on the line at infinity on any line perpendicular to λ. Thus, the line perpendicular to λ and passing through x is simply

$$(A, B, 0)^t \times x.$$

Note that $(A, B, 0) = (A, B, C) - (0, 0, C)$, so that

$$(A, B, 0) = \lambda - (\lambda \mathbf{n}^t)\mathbf{n} = \mathbf{n} \times (\lambda \times \mathbf{n})^t.$$

Thus, we may define the line through x perpendicular to λ by

$$x \perp \lambda := (\mathbf{n} \times (\lambda \times \mathbf{n})^t) \times x, \tag{8}$$

since this line passes through both $\mathbf{n} \times (\lambda \times \mathbf{n})^t$ and x.

It may also be useful to write (8) in the form

$$x \perp \lambda = (\lambda \times \mathbf{n})^t(\mathbf{n}x) - ((\lambda \times \mathbf{n})^t x)\mathbf{n}. \tag{9}$$

It is important to note here that when writing $x \perp \lambda$, it is always implicitly assumed that $\lambda \not\equiv \lambda_\infty$ and $\lambda_\infty x \neq 0$ (x does not lie on the line at infinity).

Then the foot of the perpendicular dropped from x onto λ is given by

$$(x \perp \lambda) \times \lambda = ((\mathbf{n} \times (\lambda \times \mathbf{n})^t) \times x) \times \lambda.$$

Unfortunately, it seems that the linear algebra is unwieldy at this point, and as such it would seem unlikely that involved problems involving dropping perpendiculars onto lines would prove tractable using projective tools.

4 Conic Sections

4.1 The General Form

There are many forms that the general equation of a conic sections takes in the literature. In the Euclidean case,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

and

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are popular. In the projective case,

$$ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0$$

and

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0$$

are used.

One of these forms, namely the last, lends itself to being elegantly described using tools from linear algebra, and so it will be the form we usually employ. It is easy to think of this as a Euclidean equation as well, simply thinking of x_1 as x, x_2 as y, and putting $x_3 = 1$. Moreover, with

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

along with defining

$$a_{21} := a_{12}, \quad a_{31} := a_{13}, \quad a_{32} := a_{23}$$

so that a is symmetric, we may write the equation of the conic in the economic form

$$x^t a x = 0, \tag{10}$$

again identifying 1×1 matrices with $\mathbb{R}$.

It is important to note how using homogeneous coordinates allows for this elegant expression. Alternatively, in a Euclidean world, one may interpret $x^t ax = 0$ as a conic section by imagining it as a quadric surface in three-dimensional Euclidean space and intersecting this surface with the plane $z = 1$. This interpretation will be used sparingly.

4.2 Tangents

One of the great advantages of using (10) is the way that tangents to conics may be represented. In general, we have that the line $x_0^t a$ is tangent to the conic at x_0. We can see this as follows, where we think of a tangent line as a line intersecting a curve at exactly one point.

Let x_0 be a point on the conic. Assume that $x_0^t a$ intersects the conic at some other point, say y_0. Then $x_0^t a y_0 = 0$, so that $y_0^t a x_0 = (x_0^t a y_0)^t = 0$. As a result, x_0 is on the line $y_0^t a$. But since y_0 is on the conic, $y_0^t a y_0 = 0$ as well, so y_0 is also on the line $y_0^t a$. Since x_0 and y_0 are both on the lines $x_0^t a$ and $y_0^t a$, these two lines must be same, so that $x_0^t a \equiv y_0^t a$, so that there is $k \neq 0$ with $x_0^t a = k y_0^t a$. But since a is invertible, we have $x_0^t = k y_0^t$, giving $x_0 \equiv y_0$. This contradicts the distinctness of x_0 and y_0, so that $x_0^t a$ intersects the conic at exactly one point.

This has an interesting consequence. Consider a tangent λ to the conic, so that for some x_0 on the conic,

$$\lambda \equiv x_0^t a.$$

We may suitably scale λ so that we have, in fact,

$$\lambda = x_0^t a.$$

Hence $\lambda a^{-1} = x_0^t$, so that

$$x_0 = (\lambda a^{-1})^t = a^{-1} \lambda^t.$$

This implies that

$$\lambda\, a^{-1} \lambda^t = \lambda x_0 = 0. \tag{11}$$

(11) is called the **tangential equation** for the conic, as it suggests that tangent lines may be thought of as "points" on the conic determined by a^{-1}. This may not come as a surprise when we recall the duality between points on lines in projective space; the line λ could be considered the "point" λ^t.

Said another way, consider all the lines satisfying an equation

$$\lambda\, b \lambda^t = 0.$$

Then these lines **envelope** a conic described by all points x satisfying

$$x^t b^{-1} x = 0.$$

Thus the duality between points on a conic and the envelope of lines tangent to a conic is made explicit.

4.3 Conic from Three Lines

Salmon makes consistent (and fruitful!) use of the following representation of conics (translated into the current notation): if λ, μ, and ν are lines in general position (i.e., not concurrent) and $k \in \mathbb{R}$, then

$$(\lambda x)(\mu x) = k(\nu x)^2 \tag{12}$$

is a (possibly) degenerate conic such that λ and μ are tangent to the conic and ν is the line through the points of tangency.

A simple calculation (just multiply it out!) reveals that the matrix for this conic is given by

$$a := \frac{1}{2}(\lambda^t \mu + \mu^t \lambda) - k \nu^t \nu. \tag{13}$$

Clearly $\det a = 0$ when $k = 0$, for in this case,

$$a(\lambda \times \mu) = \mathbf{0}.$$

It turns out that a is nondegenerate when $k \neq 0$. Let's take a moment to see why this is.

Assume that λ, μ, and ν are in general position (in the sense of §**2.4**), so that

$$\lambda(\mu \times \nu) = \mu(\nu \times \lambda) = \nu(\lambda \times \mu) = 1.$$

Then a generic x may be written

$$x = r(\lambda \times \mu) + s(\mu \times \nu) + t(\nu \times \lambda).$$

To see when $ax = \mathbf{0}$, note that the previous equations imply that

$$ax = \frac{1}{2}t\lambda^t + \frac{1}{2}s\mu^t - kr\nu^t.$$

Since λ, μ, and ν are in general position, this may only occur when

$$\frac{1}{2}t = 0, \quad \frac{1}{2}s = 0, \quad -kr = 0.$$

So $t = 0$, $s = 0$, and $kr = 0$. Since $x \neq \mathbf{0}$, not all of r, s, and t can be zero. Hence $k = 0$; that is, $ax = \mathbf{0}$ is possible for nonzero x only when $k = 0$. So if $k \neq 0$, a is nondegenerate.

Picking up where we left off, let's see why λ is tangent to a. Given the representation, λ is necessarily tangent to a at the point $\lambda \times \nu$. Since $\lambda \times \nu$ is on both λ and ν, it is easy to see that $x = \lambda \times \nu$ satisfies (12).

Again, since $\lambda \times \nu$ is on both λ and ν, we must have

$$\lambda(\lambda \times \nu) = 0, \quad \nu(\lambda \times \nu) = 0,$$

and consequently

$$(\lambda \times \nu)^t\lambda^t = 0, \quad (\lambda \times \nu)^t\nu^t = 0.$$

Thus, we see that

$$\begin{aligned}(\lambda \times \nu)^t a &= \frac{1}{2}((\lambda \times \nu)^t\lambda^t\mu + (\lambda \times \nu)^t\mu^t\lambda) - k(\lambda \times \nu)^t\nu^t\nu \\ &= \frac{1}{2}(\lambda \times \nu)^t\mu^t\lambda \\ &\equiv \lambda.\end{aligned}$$

But recall that this is precisely the condition that λ be tangent to a at the point $\lambda \times \nu$. Similarly, we see that μ is tangent to a at $\mu \times \nu$.

4.4 Conic from Five Points

The idea from the previous section provides a nice way to find the equation of a conic from five points, say x_1, x_2, x_3, x_4, and x_5. Clearly,

$$((x_1 \times x_2)x)((x_3 \times x_4)x) = k((x_1 \times x_3)x)((x_2 \times x_4)x)$$

passes through the first four points. As long as the five points are in general position, putting

$$k := \frac{((x_1 \times x_2)x_5)((x_3 \times x_4)x_5)}{((x_1 \times x_3)x_5)((x_2 \times x_4)x_5)}$$

ensures that it passes through x_5 as well.

4.5 Conic from Three Lines, Revisited

The equation determining a general conic passing through four points given in the previous section,

$$((x_1 \times x_2)x)((x_3 \times x_4)x) = k((x_1 \times x_3)x)((x_2 \times x_4)x),$$

may also be used to derive (12). We consider the limiting case where $x_1 \to x_2$ and $x_3 \to x_4$, where we imagine x_1, x_2, x_3, and x_4 in order along a conic.

As x_1 approaches x_2, the secant line $x_1 \times x_2$ approaches the tangent line λ at x_2. Similarly, as x_3 approaches x_4, the secant $x_3 \times x_4$ approaches the tangent μ at x_4. Of course, when both limits occur at the same time, the line $x_1 \times x_3$ approaches the line $x_2 \times x_4$, which we denote by ν, the line containing both points of tangency. This results in the equation

$$(\lambda x)(\mu x) = k(\nu x)^2,$$

in agreement with (12).

5 References

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