Chapter 9

Duality

We are now prepared to address duality, a topic central to the study of polyhedra. Our approach will not be the most general possible, but will suffice for a discussion of the Archimedean solids and their duals. Many interesting polyhedra shall be encountered during our investigation.

9.1 Basic Concepts

We begin by examining the process by which a dual polyhedron is created from its parent polyhedron. Consider first the cuboctahedron as in Figure 9.1. On each edge, find that point which, when joined to the center of the polyhedron, results in a segment perpendicular to that edge. (Note that in the case of the Archimedean solids, this is always the midpoint of the edge.) Then the line in space perpendicular to both this segment and the original edge contains an edge of the dual polyhedron. By executing this procedure at each edge of the cuboctahedron, one finds that the lines formed as perpendiculars to the edges soon begin intersecting each other at vertices of the dual polyhedron (such as a and b in Figure 9.1). The completion of the process for the cuboctahedron yields the rhombic dodecahedron of Figure 9.2, whose name is derived from the fact that each of its twelve faces is a rhombus.

In examining Figures 9.1 and 9.2, one may make several observations which happen to be valid for any Archimedean solid and its dual. For example:

1. The faces of the Archimedean dual are congruent to one another. This results from the fact that there is the same arrangement of faces at each vertex of an Archimedean solid (see §**5.1**).

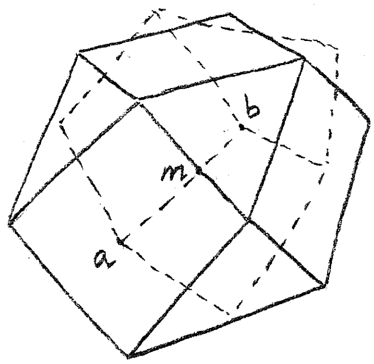


Figure 9.1

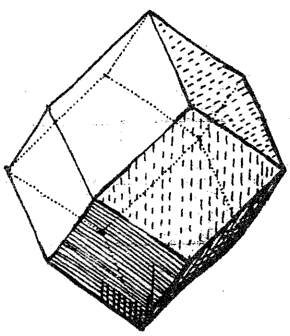


Figure 9.2

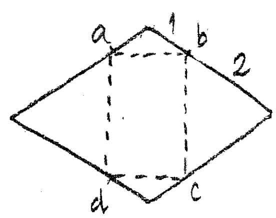
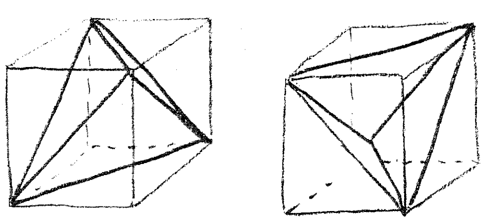
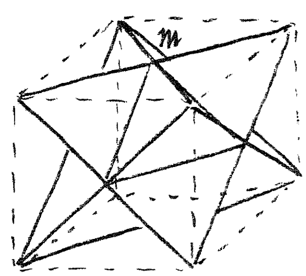


Figure 9.3



(a)



(b)

Figure 9.4

2. Upon superposing the dual on its parent (seen partially in Figure 9.1), one sees that each edge of the dual is perpendicular to an edge of the parent. Moreover, each edge of the Archimedean dual bisects the corresponding edge of its parent. The converse, however, is not true – the edges of the parent do *not* necessarily bisect those of the dual. For example, one may show that the points where the edges of the cuboctahedron meet those of the rhombic dodecahedron divide the edges of the rhombic dodecahedron in the ratio 1 : 2 (see Figure 9.3). In addition, it is these points (and not the midpoints, as was the case for the cuboctahedron) on the edges of the dual which, when joined to the center of the dual, form segments perpendicular to the edges of the dual.

3. As a result of the process for constructing the dual, we see that the number of edges on an Archimedean solid is the same as the number of edges on its dual. Moreover, it is also evident from the construction process that the number of faces of the parent is the same as the number of vertices of the dual, while dually, the number of vertices on the parent is the same as the number of faces on the dual. These "reciprocal" relationships are of the type encountered in a discussion of duality in almost any branch of mathematics.

4. It is evident from the construction process that taking the dual of an Archimedean dual results in the parent Archimedean solid. For example, the dual of the rhombic dodecahedron (itself an Archimedean dual) is in fact a cuboctahedron.

9.2 Further Examples

Before going on to discuss the calculation of edge and dihedral angles of duals, we look at several examples of duals of Archimedean solids. We first consider the tetrahedron. As is seen in Figure 9.4(a), a tetrahedron may be inscribed in a cube in two different ways; in each case the six edges of the tetrahedron consist of one diagonal from each face of the cube. Superposing these two tetrahedra yields Figure 9.4(b), where it is evident that these tetrahedra are, in fact, duals of each other. Indeed, the centers of the tetrahedron and cube coincide, and the segment joining this common center and a center of a face of the cube (such as m in Figure 9.4(b)) is orthogonal to both of the face diagonals (edges of the tetrahedra) which are perpendicular at m. Note that the edges of the dual tetrahedron are in fact bisected by those of the parent (in contrast to Observation 3 in the previous section). Such will be the case only when the parent solid is a Platonic solid.

The polyhedron in Figure 9.4(b) is often called a **stella octangula** (Exercise 6(b) of Chapter 2 and model **19** in *Polyhedron Models*), indicating that these two mutually intersecting tetrahedra may be formed by affixing a small tetrahedron to each face of an octahedron. Because the polyhedron dual to a tetrahedron is another tetrahedron, the tetrahedron is sometimes said to be **self-dual**.

What of the other Platonic solids? It is clear from Figure 9.5 [*not yet included in text*] that the octahedron is dual to the cube (and vice versa), and from Figure 9.6 [*not yet included in text*] that the icosahedron is dual to the dodecahedron (and vice versa). (See also models **43** and **47**, respectively, in *Polyhedron Models.*) In fact, due to the highly symmetrical nature of the Platonic solids, it is the case that among the group of Platonic and Archimedean solids:

1. The Platonic solids are the only solids whose duals are among the group as well, and

2. The Platonic solids are the only solids among the group all of whose dual edges are bisected by the parent solid.

For completeness, we include the duals of all the Platonic and Archimedean solids below. Names for these polyhedra are given in Table 9.1. A few explanatory remarks are in order, however.

1. A name such as "pentakis dodecahedron" indicates that this polyhedron may be formed by placing a *penta*gonal (hence "*penta*kis") pyramid on each face of a dodecahedron (here, the triangular faces of the pyramids are not necessarily equilateral).

2. Adjectives such as "rhombic," "pentagonal," and "trapezial" indicate the shape of the faces of the polyhedron. Here, "pentagonal" does not necessarily imply *regular* pentagons. A **trapezium** is a quadrilateral with no two sides parallel, hence the adjective "trapezial." Many authors use "trapezoidal" rather than "trapezial;" it happens, according to my dictionary, that the British interchange the meanings of "trapezium" and "trapezoid."

3. "Icositetra" is a prefix indicating "24," "triaconta" denotes "30," and "hexeconta" means "60." These prefixes refer to the number of faces on the polyhedron.

Polyhedron	Dual Polyhedron
Tetrahedron	Tetrahedron
Cube	Octahedron
Octahedron	Cube
Icosahedron	Dodecahedron
Dodecahedron	Icosahedron
Truncated tetrahedron	Triakis tetrahedron
Truncated octahedron	Tetrakis hexahedron
Truncated icosahedron	Pentakis dodecahedron
Truncated cube	Triakis octahedron
Truncated dodecahedron	Triakis icosahedron
Cuboctahedron	Rhombic dodecahedron
Icosidodecahedron	Rhombic triacontahedron
Rhombitruncated cuboctahedron	Hexakis octahedron
Rhombicuboctahedron	Trapezial icositetrahedron
Rhombitruncated icosidodecahedron	Hexakis icosahedron
Rhombicosidodecahedron	Trapezial hexecontahedron
Snub cube	Pentagonal icositetrahedron
Snub dodecahedron	Pentagonal hexecontahedron

Table 9.1

9.3 Edge Angles

How can the edge angles of an Archimedean dual be calculated from information about the parent Archimedean solid? This section will focus on answering that question.

Let us consider for a moment Figure 9.7(a), where we see a truncated tetrahedron and a face rtu of its dual, a triakis tetrahedron. Here, O is the center of the truncated tetrahedron, p is the center of a hexagonal face of the truncated tetrahedron, and q is the center of a triangular face. Because of the symmetry of the truncated tetrahedron about this triangular face, it follows that the vertex of the triakis tetrahedron above this triangular face – labelled r in Figure 9.7(a) and formed as the intersection of the perpendicular bisectors of the edges of this face created in the process of dualizing the truncated tetrahedron – lies on the ray $\overrightarrow{Oq}$. For similar reasons, the vertex t of the triakis tetrahedron above the hexagonal face of which p is the center must lie on the ray $\overrightarrow{Op}$. Finally, s is the point on the edge rt of the triakis tetrahedron which is the midpoint of the edge of the truncated tetrahedron in which it lies, and hence the segment Os is perpendicular to the edge rt.

That part of a planar cross-section of Figure 9.7(a) containing O, p, q, r, s, and t is shown in Figure 9.7(b). Since s is the midpoint of the edge incident to the faces of which p and q are centers, it follows that $\angle qsp$ has the same measure as the dihedral angle $\mathsf{D}^t_{3,3}$ of the truncated tetrahedron. Again, due to the symmetry of the truncated tetrahedron, the segment Op is perpendicular to the hexagonal face of which p is the center. As a result, we see that segment Op and ps in Figure 9.7(b) are perpendicular. Similarly, Oq and qs are also perpendicular. Finally, since rt is an edge of the triakis tetrahedron (whose center is O), we see that $\angle rOt$ is an edge angle of the triakis tetrahedron, which shall be denoted by $\bar{\mathsf{E}}^t_{3,3}$.

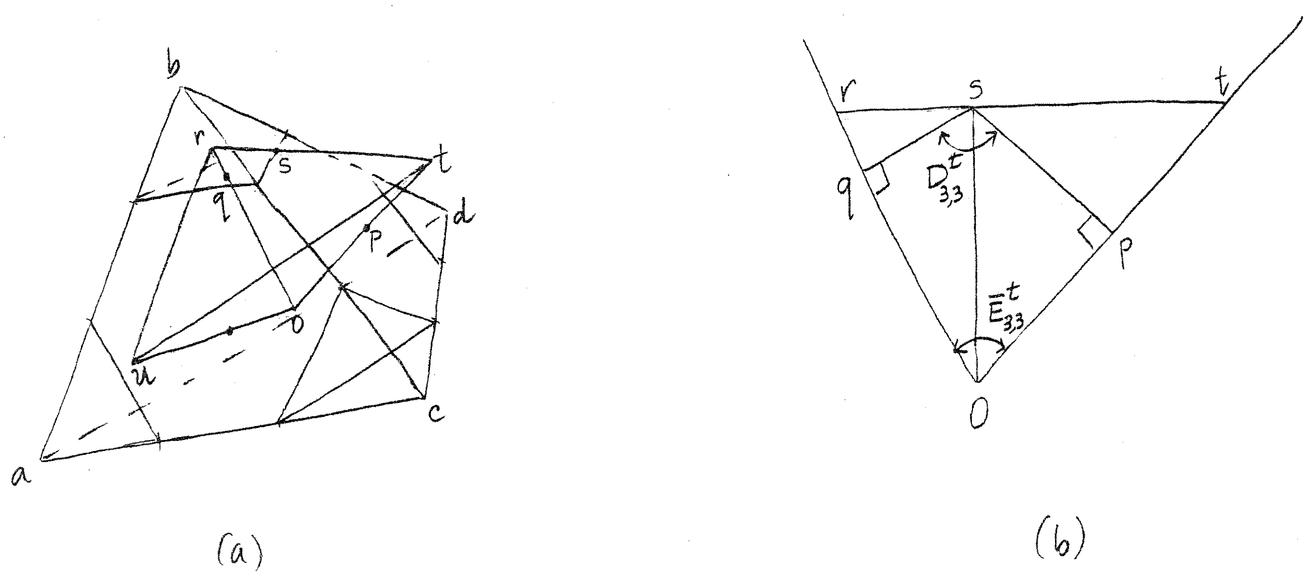


Figure 9.7

It is evident from examining the quadrilateral $Oqsp$ that $\mathsf{D}^t_{3,3}$ and $\bar{\mathsf{E}}^t_{3,3}$ are supplementary, so that $\mathsf{D}^t_{3,3} + \bar{\mathsf{E}}^t_{3,3} = \pi$, and hence $\cos \bar{\mathsf{E}}^t_{3,3} = -\cos \mathsf{D}^t_{3,3}$. This phenomenon is not peculiar to this pair of duals; in general, we see that

> (D_1) *The edge angle subtended by an edge of an Archimedean dual is supplementary to that dihedral angle of the Archimedean solid whose edge is bisected by the dual edge.*

This relationship suggests the notation $\bar{\mathsf{E}}^t_{3,3}$ for the edge angle of the triakis tetrahedron subtended by the edge rt. Thus, if $\mathsf{D}^*_{p,q}$ represents a dihedral angle of an Archimedean solid, then $\bar{\mathsf{E}}^*_{p,q}$ represents the corresponding edge angle of its dual, and we have $\cos \bar{\mathsf{E}}^*_{p,q} = -\cos \mathsf{D}^*_{p,q}$. We see, then, that $\bar{\mathsf{E}}_{3,3}$ is the edge angle of the edge tu of the triakis tetrahedron (see Figure 9.7(a)) which intersects an edge of the truncated tetrahedron at which the dihedral angle is $\mathsf{D}_{3,3}$, the same as that of the regular tetrahedron. It follows, therefore, that tu would also be an edge of the tetrahedron dual to the tetrahedron whose vertices are a, b, c, and d. This gives the triakis tetrahedron its name, for it may be imagined as a tetrahedron with squat triangular ("triakis") pyramids affixed to its faces.

We now proceed to investigate Figure 9.7(b) in more detail. Upon consideration of the similar right triangles ΔOqs and ΔOsr, we see that $[Oq]/[Os] = [Os]/[Or]$; likewise, we note that the similarity of right triangles ΔOps and ΔOst implies that $[Op]/[Os] = [Os]/[Ot]$. These two relationships together imply that

$$[Oq][Or] = [Os]^2 = [Op][Ot].$$

Such relationships can be written for any Archimedean solid and its dual; they can be imagined as a result of creating Archimedean duals by the process of **polar reciprocation**, a topic which will not be addressed here (but see Wenninger's *Dual Models,* pp. 1-5).

9.4 Faces of Archimedean Duals

Upon looking at the triakis tetrahedron, for example, one observes that the faces are not equilateral triangles – in fact, far from it! How can we learn more about the shape of these triangular faces? We will investigate a useful procedure called the "Dorman Luke construction" by Wenninger in *Dual Models* (p. 30). (Actually, this procedure derives from the process of polar reciprocation.)

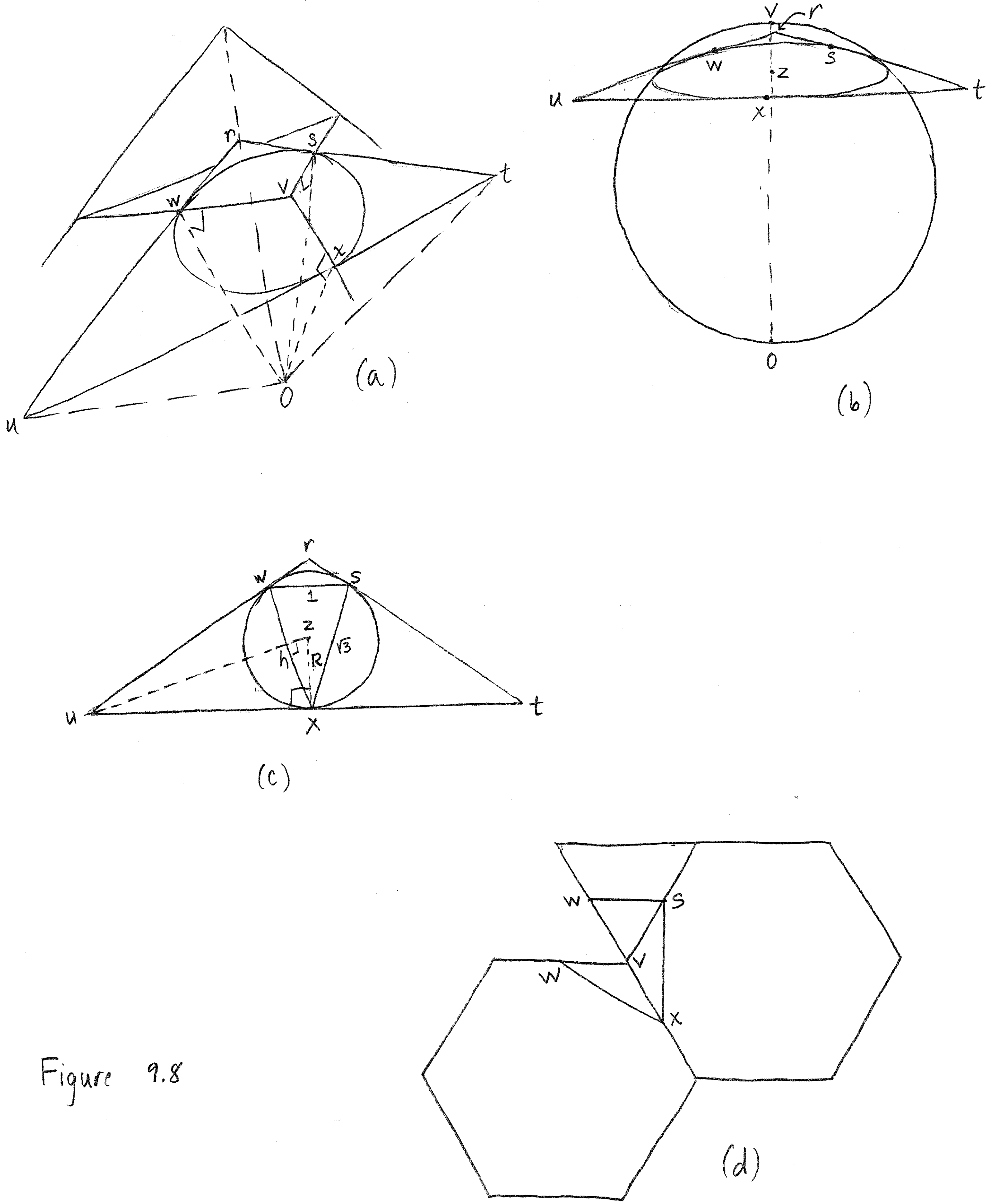


Figure 9.8

We begin with an expanded view of the face rtu of the triakis tetrahedron, as in Figure 9.8(a). Here, in addition to the points labelled in Figure 9.7(a), are the midpoints w and x of two other edges of the truncated tetrahedron, and a vertex v of the truncated tetrahedron. As a result of the dualizing process, we see that triangles ΔOsv, ΔOwv, and ΔOxv are all right triangles.

Since these triangles all share the same hypotenuse Ov, and since any triangle inscribed in a semicircle is a right triangle, it follows that these triangles may be inscribed in a sphere with diameter Ov (see Figure 9.8(b)). Moreover, since $[vs] = [vw] = [vx]$, this common value being one-half the edge length of the truncated tetrahedron, it follows that s, w, and x all lie on a circle which is the intersection of the sphere and a plane which is perpendicular to Ov (intersecting Ov at z) and which passes through s, w, and x. This circle, whose center is z, is shown in Figure 9.8(c). Because of the perpendicularity of the dual edges and the parent edges, we see that the sides of the face rtu of the triakis tetrahedron are in fact tangent to this circle at s, x, and w. Note that the lengths of ws, sx, and xw may be easily determined as s, w, and x are the midpoints of edges of the truncated tetrahedron (a flattened vertex of which is shown in Figure 9.8(d)). The triangle Δswx is called a **vertex figure** of the truncated tetrahedron.

To determine the shape of Δrtu, we assume for specificity that $[ws] = 1$, and hence from Figure 9.8(d) that $[wx] = [xs] = \sqrt{3}$. It may be shown that given a triangle whose sides have length a, b, and c, then the radius R of the circumcircle (i.e., the circle circumscribing the triangle) satisfies

$$R^2 = \frac{a^2b^2c^2}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

With $a = b = \sqrt{3}$ and $c = 1$, we find that $R^2 = 9/11$ and hence $R = [zx] = 3/\sqrt{11}$. It follows (see Figure 9.8(c)) that

$$\cos(\angle zux) = \sin(\angle hzx) = \frac{[hx]}{R} = \frac{\sqrt{3}/2}{3/\sqrt{11}} = \frac{\sqrt{11}}{2\sqrt{3}},$$

and hence

$$\cos(\angle rut) = 2\cos^2(\angle zux) - 1 = 2 \cdot \frac{11}{12} - 1 = \frac{5}{6}.$$

One may similarly conclude that $\cos(\angle urt) = -7/18$.

We may summarize our observations succinctly as follows:

(D_2) *The sides of a face of an Archimedean dual are tangent to the circumcircle of a vertex figure of the Archimedean parent at the vertices of this figure.*

We remark in general that a vertex figure of an Archimedean solid is obtained by selecting a vertex of the solid and then moving along each edge incident at that vertex the same distance in order to obtain the vertices of the vertex figure. Thus, as a further example, the vertex figure for a cuboctahedron is a $\sqrt{2} : 1$ rectangle, and thus a face of its dual rhombic dodecahedron may be found as in Figure 9.9.

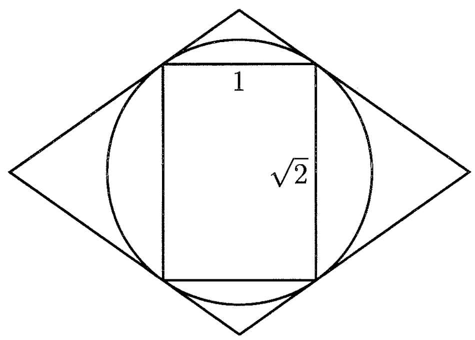


Figure 9.9

Although the construction (D_2) of the faces of Archimedean duals is geometrically intriguing and involves many of the important relationships between the Archimedean solids and their duals, it is a somewhat arduous task in an algebraic sense. A simpler algebraic method will be explored in §**9.6**.

9.5 Dihedral Angles

We wish to extend our analysis of the previous section in order to calculate dihedral angles of Archimedean duals. We will soon discover, happily, that these calculations are rather simple, thereby revealing a straightforward method for finding the angles of the faces of the duals.

Let us first attempt to find the dihedral angle at the edge tu of the triakis tetrahedron in the previous section (see Figure 9.8(a)–(c)). To do so, we consider the other face of the triakis tetrahedron of which tu is an edge (see Figure 9.10(a)), where r' is the other vertex of that face and z' is the center of the circle to which the sides of $\Delta r'tu$ are tangent.

A cross-section of Figure 9.10(a) is shown in Figure 9.10(b). Here, v is the vertex of the truncated tetrahedron on the ray $\overrightarrow{Oz}$ (see Figure 9.8(b)), and similarly, v' is the vertex of the truncated tetrahedron on the ray $\overrightarrow{Oz'}$. Recall (Figures 9.8(b),(c)) that Ov is perpendicular to the plane containing the circle of which z is the center, so that $\angle Ozx$ must be a right angle; one similarly argues that $\angle Oz'x$ must likewise be a right angle. Since a tangent to a circle is perpendicular to a radius drawn to the point of tangency, it follows that both zx and $z'x$ are perpendicular to tu (as in Figure 9.10(a)), and therefore $\angle zxz'$ (see Figure 9.10(b)) must have the same measure as the dihedral angle of the triakis tetrahedron, which we denote by $\bar{\mathsf{D}}^t_{3,3}$. Since O is the center of the truncated tetrahedron as well, we see that $\angle vOv'$ has measure $\mathsf{E}^t_{3,3}$, the edge angle of the truncated tetrahedron.

These considerations lead us, upon examining the quadrilateral $z'Ozx$ in Figure 9.10(b), to the conclusion that $\mathsf{E}^t_{3,3}$ and $\bar{\mathsf{D}}^t_{3,3}$ are supplementary; i.e., that $\mathsf{E}^t_{3,3} + \bar{\mathsf{D}}^t_{3,3} = \pi$. But had we considered the dihedral angle at the edge rt instead, we would nonetheless have arrived at a figure similar to Figure 9.10(b), and would have inevitably concluded that this dihedral angle was supplementary to $\mathsf{E}^t_{3,3}$. We make the following generalization:

(D_3) *The dihedral angles of an Archimedean dual all have the same measure, all being supplementary to the edge angle of the parent Archimedean solid.*

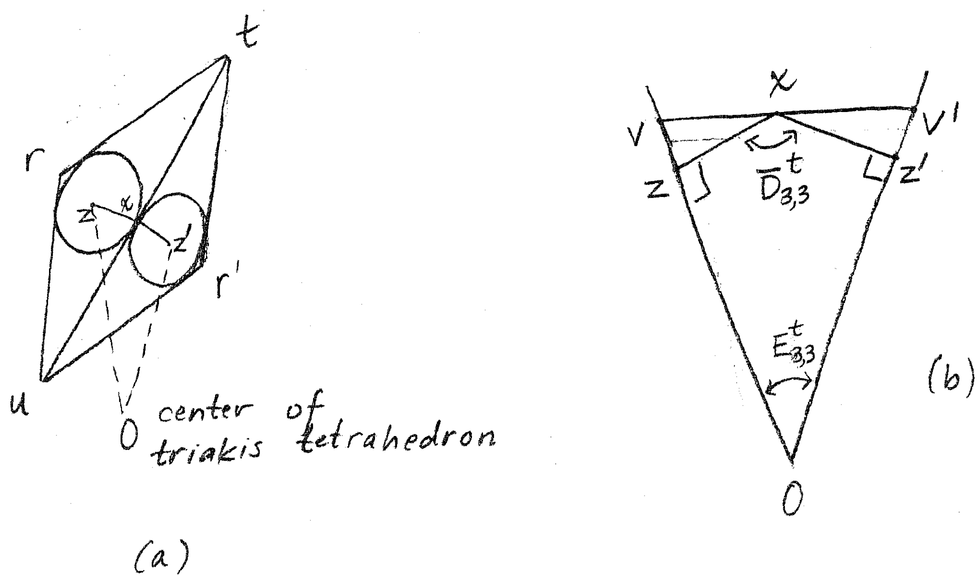


Figure 9.10

We use similar notation for the other duals; therefore, if $\mathsf{E}^*_{p,q}$ represents the edge angle of some Archimedean solid, then $\bar{\mathsf{D}}^*_{p,q}$ represents the dihedral angle of its dual, so that $\mathsf{E}^*_{p,q} + \bar{\mathsf{D}}^*_{p,q} = \pi$ and hence $\cos \bar{\mathsf{D}}^*_{p,q} = -\cos \mathsf{E}^*_{p,q}$.

9.6 Faces Revisited

Since all of the dihedral angles of the triakis tetrahedron have the same measure, and since all the angles of faces incident at the same vertex have the same measure, calculations of the facial angles is a straightforward application of (3.10).

Consider, as an example, the acute angle φ (such as $\angle rut$ in Figure 9.8(c)) of the faces of the triakis tetrahedron. Solving (3.10) for φ yields

$$\cos\varphi = 2\frac{1+\cos\frac{2\pi}{q}}{1-\cos\mathsf{D}} - 1. \tag{9.1}$$

In our case, $q = 6$ since six acute angles meet at each of four vertices of the triakis tetrahedron, and of course $\mathsf{D} = \bar{\mathsf{D}}^t_{3,3}$. These values may be substituted into (9.1) (where Table 6.1 is used to find $\cos \bar{\mathsf{D}}^t_{3,3} = -\cos \mathsf{E}^t_{3,3}$), which results in $\cos\varphi = 5/6$.

If φ represents the obtuse angle of the face of the triakis tetrahedron (such as $\angle urt$ in Figure 9.8(c)), then as three of these angles meet at each of four vertices of the triakis tetrahedron, we find upon using (9.1) with $q = 3$ that $\cos\varphi = -7/18$.

Values of φ for the remainder of the Archimedean duals may be found in Table 9.2.

Polyhedron	q	$\cos\varphi$	φ (deg)
Triakis Tetrahedron	3	$-\frac{7}{18}$	112.885
	6	$\frac{5}{6}$	33.557
Tetrakis Hexahedron	4	$\frac{1}{9}$	83.621
	6	$\frac{2}{3}$	48.190
Pentakis Dodecahedron	5	$\frac{9\sqrt{5}-7}{36}$	68.619
	6	$\frac{9-\sqrt{5}}{12}$	55.691

Table 9.2

Polyhedron	q	$\cos\varphi$	φ (deg)
Triakis Octahedron	3	$\frac{1-2\sqrt{2}}{4}$	117.201
	8	$\frac{2+\sqrt{2}}{4}$	31.400
Triakis Icosahedron	3	$-\frac{3}{10}\tau$	119.039
	10	$\frac{1}{10}(7+\tau)$	30.480
Rhombic Dodecahedron	3	$-\frac{1}{3}$	109.471
	4	$\frac{1}{3}$	70.529
Rhombic Triacontahedron	3	$-\frac{1}{\sqrt{5}}$	116.565
	5	$\frac{1}{\sqrt{5}}$	63.435
Hexakis Octahedron	4	$\frac{2-\sqrt{2}}{12}$	87.202
	6	$\frac{6-\sqrt{2}}{8}$	55.025
	8	$\frac{1+6\sqrt{2}}{12}$	37.773
Hexakis Icosahedron	4	$\frac{5-2\sqrt{5}}{30}$	88.992
	6	$\frac{15-2\sqrt{5}}{20}$	58.238
	10	$\frac{9+5\sqrt{5}}{24}$	32.770
Trapezial Icositetrahedron	3	$-\frac{2+\sqrt{2}}{8}$	115.263
	4	$\frac{2-\sqrt{2}}{4}$	81.579
Trapezial Hexacontahedron	3	$-\frac{5+2\sqrt{5}}{20}$	118.269
	4	$\frac{5-2\sqrt{5}}{10}$	86.974
	5	$\frac{9\sqrt{5}-5}{40}$	67.783
Pentagonal Icositetrahedron	3	−0.419643	114.812
	4	0.160713	80.752
Pentagonal Hexecontahedron	3	−0.471576	118.137
	5	0.383433	67.453

Table 9.2 (continued)

9.7 Exercises

1. With the nets provided at the end of the chapter, build the rhombic dodecahedron, the triakis octahedron, and the trapezial icositetrahedron.

2. As in Figure 9.11, we see that a triakis tetrahedron may be imagined as a regular tetrahedron with four squat tetrahedra affixed to its faces. Here, e is on df and is the center of a triangular face of the regular tetrahedron. In this exercise, we wish to find the ratio $[ed]/[ef]$; that is, the ratio of the height of the regular tetrahedron to the height of the squat tetrahedra. Assume that the edges of the tetrahedron have length 2, so that $[cd] = 2$.

   (a) Using data from Table 9.2, show that
   $$[cf] = \frac{6}{5}.$$

   (b) Find $[ce]$, and then by considering right triangles Δced and Δcef, show that
   $$\frac{[ed]}{[ef]} = 5.$$

   (c) Using (b), show that the ratio of the volume of the triakis tetrahedron relative to its base tetrahedron is 9/5.

   (d) Show also that $\angle dcf$ has the same measure as $D_{3,3}$.

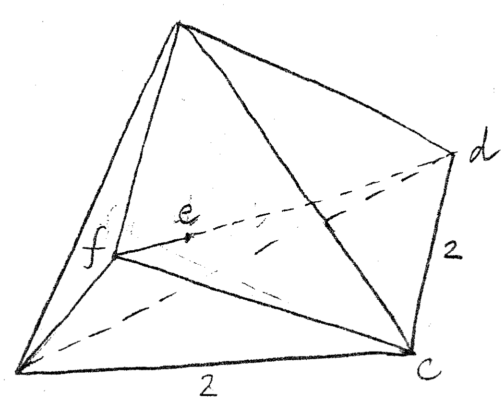


Figure 9.11

3. State and demonstrate results analogous to those of the previous exercise for the triakis octahedron and the triakis icosahedron.

4. A tetrakis hexahedron may be imagined as a cube with six square pyramids affixed to its faces. Using a procedure analogous to (a)–(c) of Exercise 2, show that the ratio of the volume of the tetrakis hexahedron to that of its base cube is $\frac{3}{2}$.

5. A rhombic dodecahedron may also be imagined as a cube with six square pyramids affixed to its faces. Describe this relationship, then calculate that the ratio of the volume of the rhombic dodecahedron to that of its base cube is 2.

6. Consider the faces of a rhombic dodecahedron, and imagine inscribing a square in each rhombus, as in Figure 9.12. Using data from Table 9.2, show that this may be accomplished by dividing the edges of each rhombus in the ratio

$$\frac{y}{x} = \sqrt{2}.$$

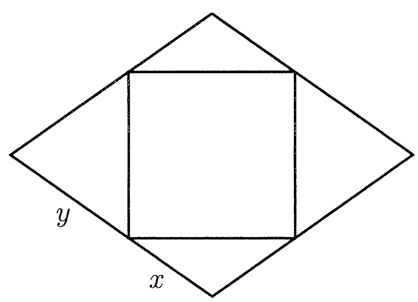


Figure 9.12

(Note: The squares thus formed are twelve square faces of the rhombicuboctahedron, so that the procedure just described shows that a rhombicuboctahedron may be inscribed in a rhombic dodecahedron.)

7. Show the following result, which is a generalization of the result in the previous exercise: a square may be inscribed in a rhombus (see Figure 9.13) by dividing the edges of the rhombus in the ratio

$$\frac{y}{x} = \cot \frac{A}{2},$$

where A is the smaller of the interior angles of the rhombus.

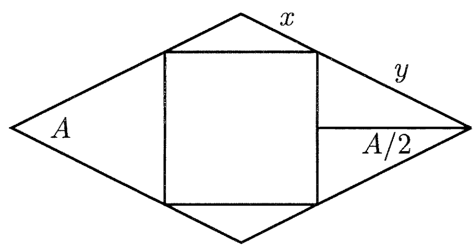


Figure 9.13

(a) What is this ratio for a rhombic triacontahedron?

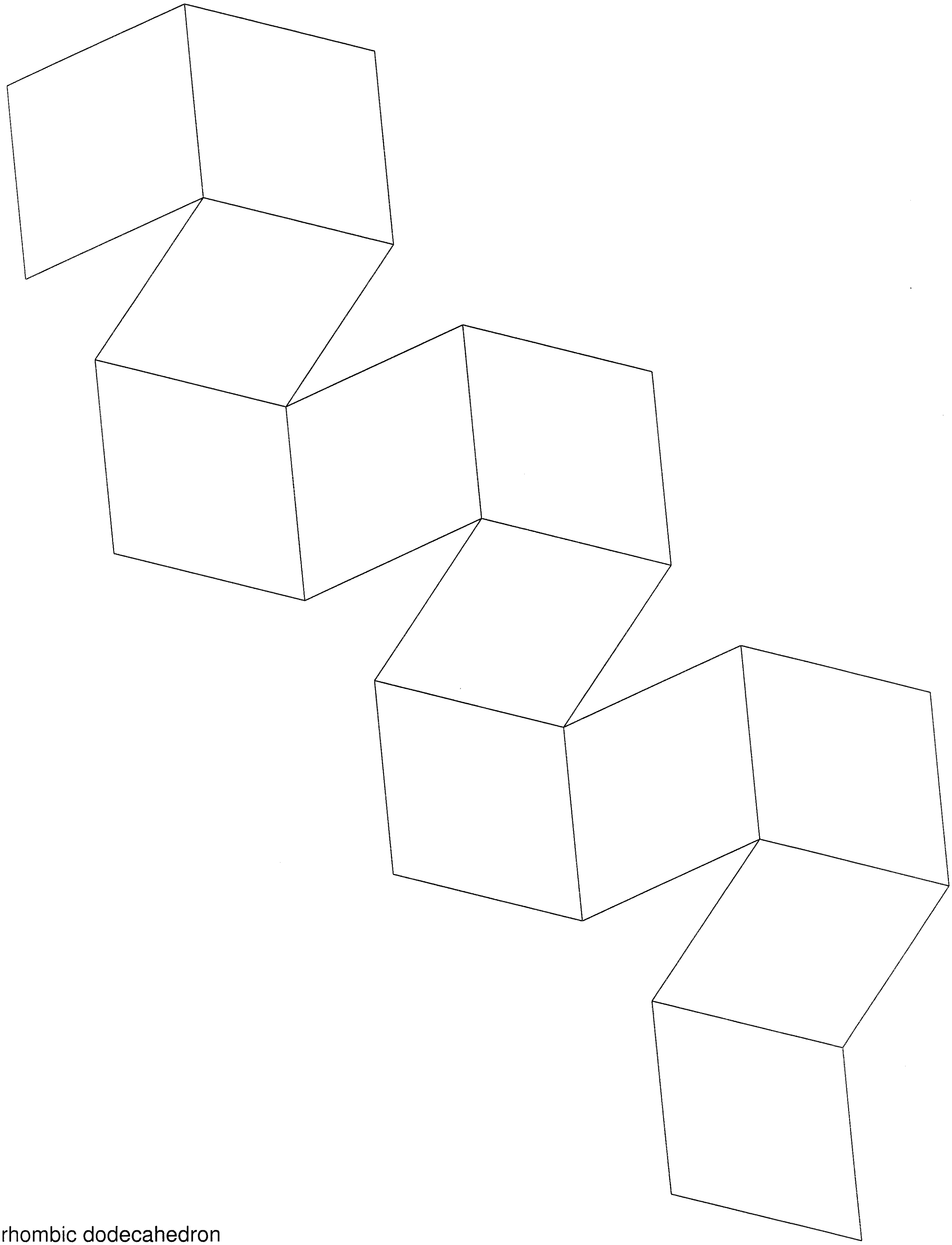
(b) As a result, what Archimedean solid may be inscribed in a rhombic triacontahedron using this procedure?

8. In this exercise, we look at at alternative way of deriving (8.24). To do so, we consider the dual of our snub polyhedron. Since our snub polyhedron has four triangles and one p-gon at each vertex, its dual has faces which are pentagons – at four of its vertices, three faces meet, while p meet at the remaining vertex. Let φ_3 denote the interior angles at the vertices of the pentagon where three faces meet, and let φ_p denote the interior angle at the remaining vertex.

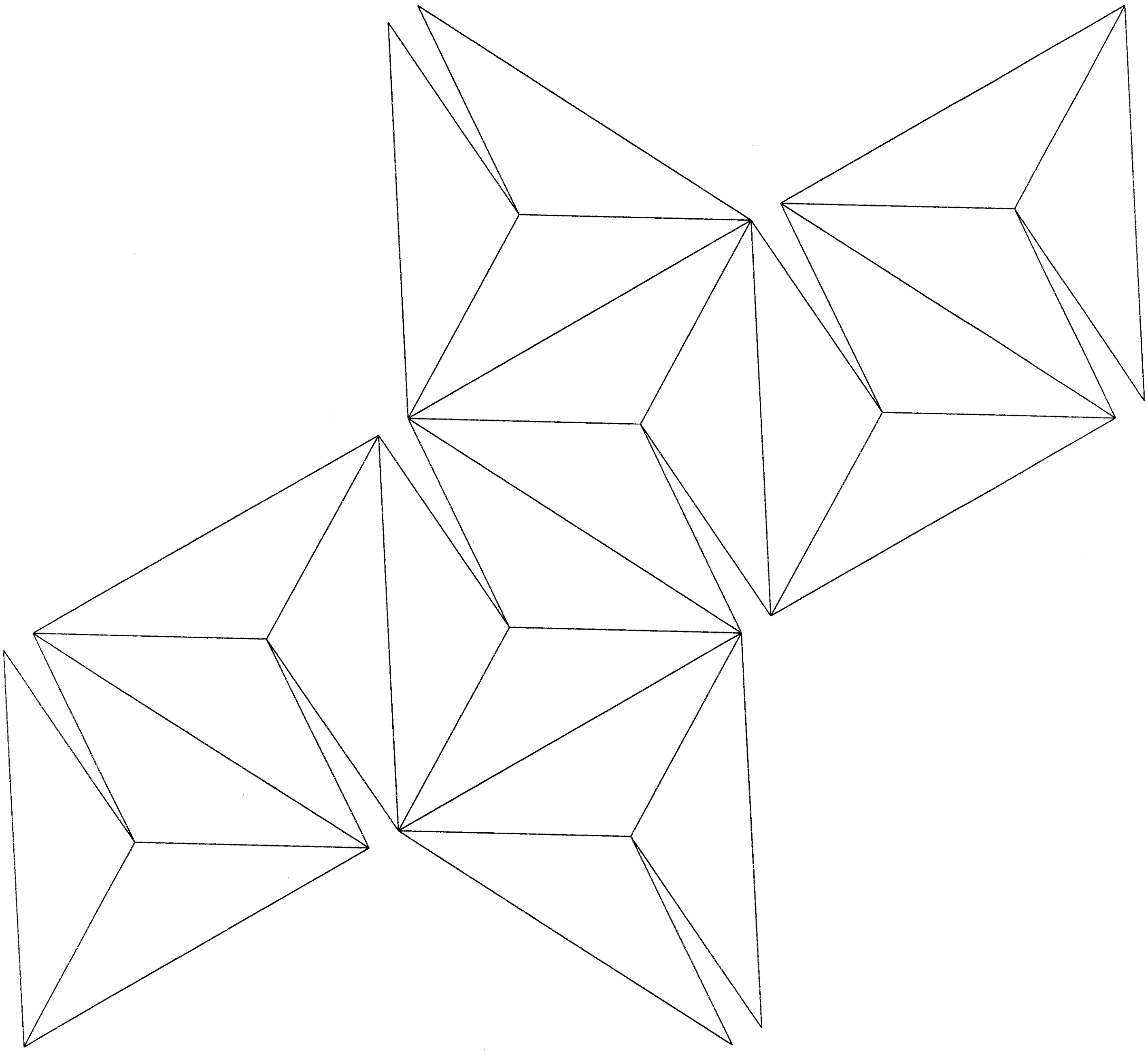
   Since the interior angles of a convex pentagon sum to 3π, it is evident that $4\varphi_3 + \varphi_p = 3\pi$, so that

$$\cos 4\varphi_3 = \cos(3\pi - \varphi_p) = -\cos\varphi_p.$$

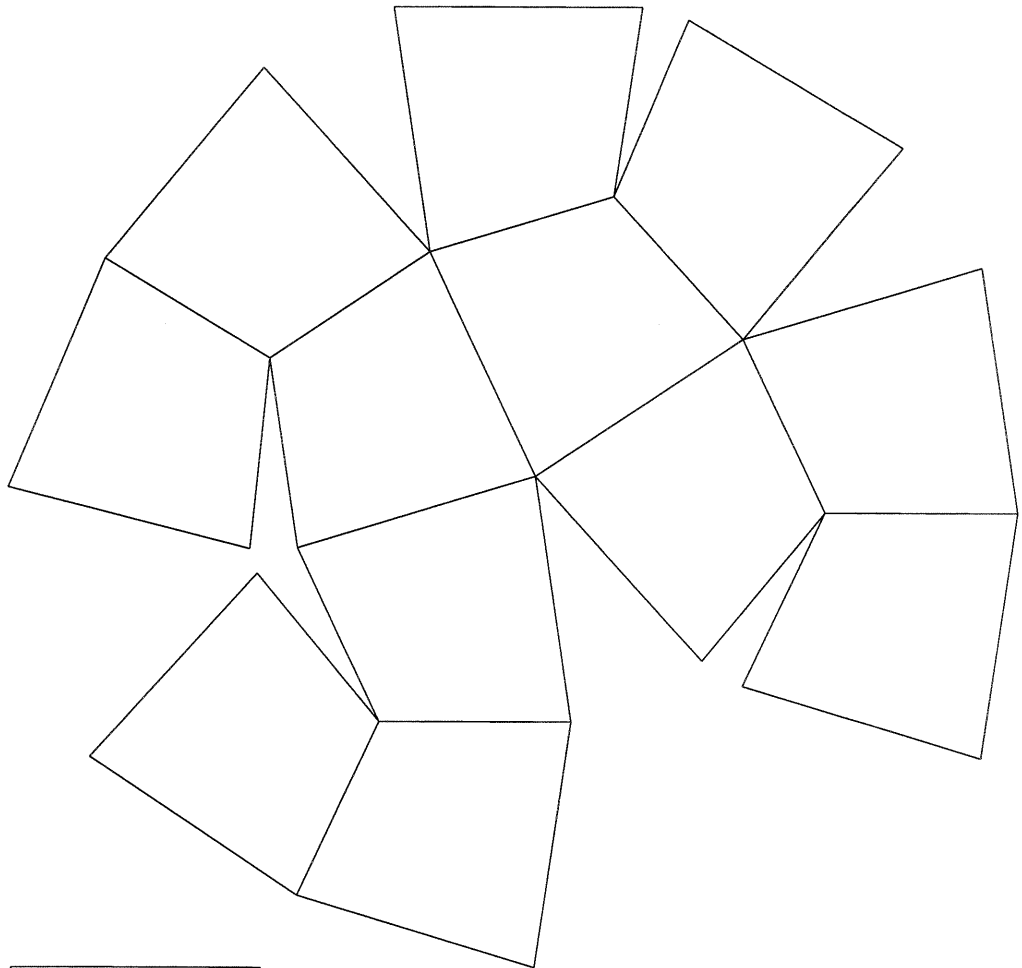
   Substitute for $\cos\varphi_3$ and $\cos\varphi_p$ from (9.1) into this expression, and thus produce an alternate derivation of (8.28).

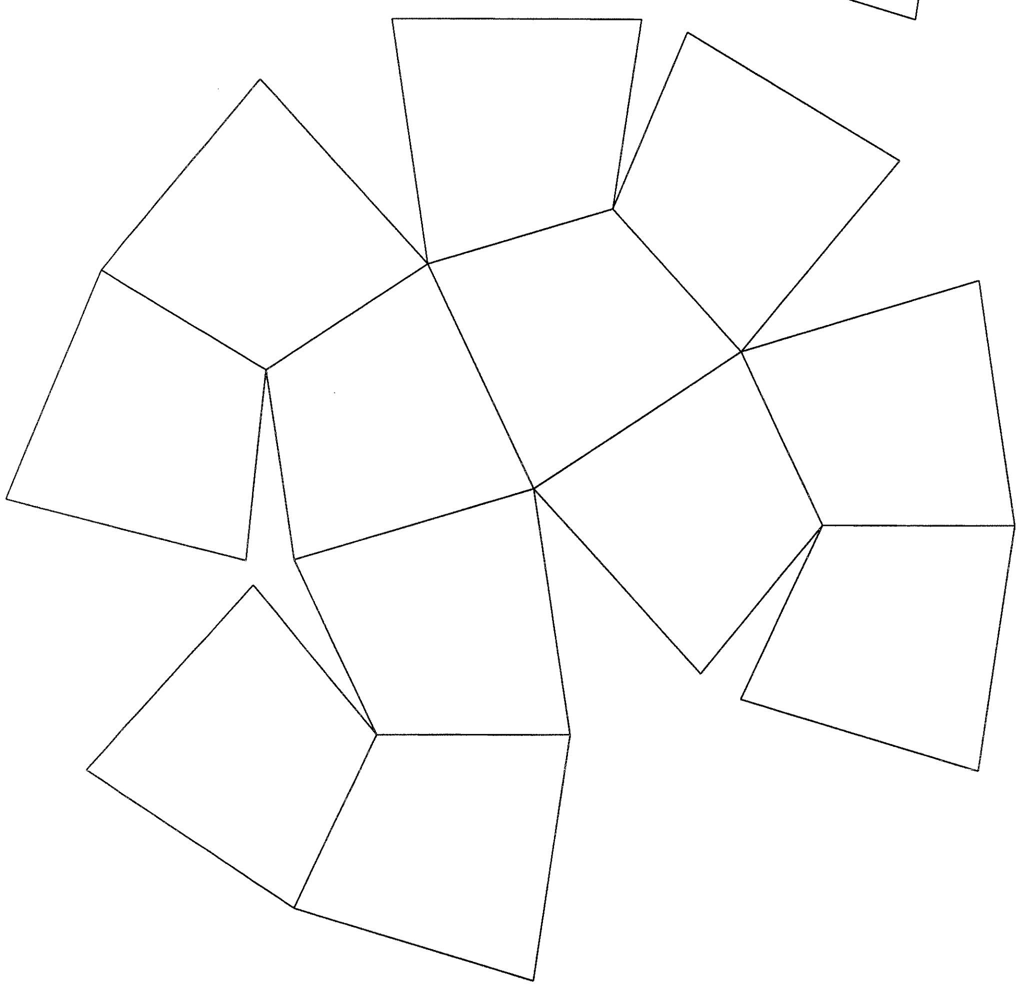


rhombic dodecahedron



triakis octahedron





trapezial icositetrahedron