Chapter 15

Matrices and Symmetry Groups

15.1 Introduction

As suggested in the previous chapter, efficient rendering of polyhedral images requires more than a knowledge of coordinates for the vertices of polyhedra. Familiarity with matrices and symmetry groups is extremely useful.

Matrices are typically dicussed in an undergraduate course in linear algebra, while symmetry groups are usually presented in an abstract algebra course. Only the barest introduction to these topics can be given in a single chapter. The interested reader should be cautioned, though, that many textbooks in linear algebra approach the topic of matrices from a purely algebraic perspective. In such texts, motivation for studying matrices is the solution of simultaneous linear equations in several variables. The reader should look for a text, such as Banchoff and Wermer's *Linear Algebra Through Geometry* (ISBN 0-387-97586-1), which considers matrices from a geometric viewpoint.

15.2 Definitions and Notations

Let $\mathbb{R}^3$ denote the set of all possible coordinate lists consisting of three real numbers. Recall that in the last chapter, care was taken to distinguish a point from its coordinates. Indeed, two coordinate systems may be in use at the same time, such as when considering an equation for a plane curve in both Cartesian and polar coordinates.

In this discussion, the only coordinate system under consideration is the Cartesian coordinate system in three dimensions. So rather than say that the point p has coordinates $(1, -1, -1)$, we may simply refer to the *point* $(1, -1, -1)$.

Before discussing matrices in particular, we must look at functions on $\mathbb{R}^3$ in general. For example, consider the function $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{A}((x, y, z)) = (x, y, 0).$$

This function takes a point and changes its z-coordinate to 0. Although defined algebraically, this function also has a geometric interpretation. **A** **projects** a point onto the xy-plane. In other words, if a point p is moved towards the xy-plane along a path parallel to the z-axis, it would eventually intersect the xy-plane at the point $\mathbf{A}(p)$.

Many other functions on $\mathbb{R}^3$ are possible, such as:

$$\mathbf{B}((x, y, z)) = (x^2, y^2, z^2), \tag{15.1}$$

$$\mathbf{C}((x, y, z)) = (x + y, y - z, x), \tag{15.2}$$

$$\mathbf{D}((x, y, z)) = (0, 0, 0), \tag{15.3}$$

$$\mathbf{E}((x, y, z)) = (\cos x, \sin z, e^y). \tag{15.4}$$

In a moment, we will look at those functions which may be represented by a **matrix**.

A **linear combination** of the variables x, y, and z is an expression of the form

$$ax + by + cz;$$

that is, the variables are simply multiplied by numbers and then added together. Some of the numbers may be negative or zero, so that

$$y - z$$

is also a linear combination of x, y, and z.

However,

$$x^2 + 4y - \pi z$$

is not, since the x is squared. The variables may only be multiplied by numbers, and may not appear as arguments to any other function.

We say that a function $\mathbf{F}$ is **linear** if given a point p, the coordinates of $\mathbf{F}(p)$ are linear combinations of the coordinates of p. Thus, the functions **A**, **C**, and **D** described earlier are linear, while **B** and **E** are not.

Note that $0 = 0x + 0y + 0z$ is a linear combination of x, y, and z, but that 1 is not. For no choice of real numbers a, b, and c does the equation

$$ax + by + cz = 1$$

hold for *all* values of x, y, and z.

A linear function $\mathbf{F}$ on $\mathbb{R}^3$ is often represented by a 3×3 matrix whose first row consists of the coefficients of x, y, and z, respectively, in the first coordinate of $\mathbf{F}((x, y, z))$, whose second row consists of the coefficients of x, y, and z, respectively, in the second coordinate of $\mathbf{F}((x, y, z))$, and whose third row is likewise defined. With this notation in mind, the function $\mathbf{C}$ described above may be represented as

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

When using matrix notation, it is customary to write the coordinates of a point in a column, as in

$$\mathbf{C}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y-z \\ x \end{pmatrix}.$$

Recall that this is a special case of an operation often referred to as **matrix multiplication**. In the general case, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}. \tag{15.5}$$

The linear functions $\mathbf{A}$ and $\mathbf{D}$ above are represented by the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\mathbf{I}$ defined by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is called the **identity matrix**; it has the property that for any point p, $\mathbf{I}(p) = p$.

The purist may remark that a matrix simply *represents* a function and is not a function itself. As it turns out, if the underlying coordinate system changes, the matrix representing a linear function also changes. But since we will not be deviating from the use of our three-dimensional Cartesian coordinate system, we will not need to notationally distinguish between a linear function and its matrix.

15.3 Matrices and Symmetry

Certain types of matrices are especially relevant to a study of polyhedra. We begin with an example.

Consider the matrix

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now consider the effect of $\mathbf{R}$ on the vertices of the cube described in §**14.2**:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{R}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{R}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

By examining Figure 14.3, it is apparent that $\mathbf{R}$ is a clockwise rotation through 120° about an axis through the center of the cube and the vertex $(1,1,1)$. Since the axis also passes through $(-1,-1,-1)$, the vertices $(1,1,1)$ and $(-1,-1,-1)$ are not moved by $\mathbf{R}$:

$$\mathbf{R}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{R}\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

We also see that

$$\mathbf{R}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{R}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{R}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus **R** takes each vertex of the cube into another, not necessarily distinct, vertex of the cube.

The rotation **R** is said to be a **symmetry** of the cube in that if we begin with the cube as described in §**14.2** and apply the linear function **R**, the resulting cube is indistinguishable from the original in that it occupies the same region of space.

Another way to say this is that if we begin with a list of the eight vertices of the cube and apply **R** to each one, we get the same eight vertices, although usually in a different order. Thus, **R** is said to be a **permutation** of the vertices of the cube. It turns out that every symmetry of the cube produces a corresponding permutation of its vertices. However, not every permutation of the cube's vertices yields a symmetry of the cube; for example, a permutation which merely switches the ends of one edge of the cube and leaves the others intact does not yield a symmetry of the cube. For a thorough discussion of the theory of permutations, consult a textbook on abstract algebra.

Another symmetry of the cube is given by

$$\mathbf{S} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By examining the effect of **S** on the vertices of the cube, we see that **S** corresponds to a 90° counterclockwise rotation around the x-axis (as viewed from the positive z-axis).

The reader might pause for a few moments and try to discover some other symmetries of the cube and their corresponding matrices. For the impatient reader, we will answer the following question: how many symmetries of the cube are there?

Let us begin by counting symmetries such as **R** described above; that is, a rotation around an axis which contains a diagonal of the cube. If we imagine holding the opposite vertices of a diagonal of a cube with our thumb and middle finger, we may physically rotate the cube in the manner described by **R**. Performing this rotation again, we obtain a 240° rotation about the axis, or the symmetry $\mathbf{R} \circ \mathbf{R} = \mathbf{R}^2$. (It is customary, when working with matrices, to denote repeated composition by the appropriate exponent and call the composition of matrices **matrix multiplication**.) If we perform this rotation again, a 360° rotation about the axis is obtained; but this does not in fact move the cube at all, since every vertex ends up exactly where it started. This may be described by the relationship

$$\mathbf{R} \circ \mathbf{R} \circ \mathbf{R} = \mathbf{R}^3 = \mathbf{I}.$$

Thus, each diagonal of the cube generates two symmetries of the cube. Since there are four diagonals, we have found eight symmetries of the cube so far.

Technically, the identity, **I**, which does not move the cube at all, is considered a symmetry of the cube. We will count it, however, only at the end. If we counted it as being generated by a rotation about an axis, it would be overcounted several times.

We may realize the rotation **S** described above by holding a cube so that our thumb and middle finger touch the centers of two opposite faces of the cube. If **S** describes a 90° counterclockwise rotation about the axis passing through these facial centers, $\mathbf{S} \circ \mathbf{S} = \mathbf{S}^2$ describes a 180° rotation about this axis, and $\mathbf{S}^3$ described a 270° rotation. Of course $\mathbf{S}^4 = \mathbf{I}$, since a 360° rotation is obtained. As we are not counting **I**, three rotations are thus produced. Since three rotations are produced by each of the three pairs of opposite faces, nine more rotations bring the count to 17.

By holding a cube by the midpoints of a pair of opposite edges, a new rotation is generated. As this must be a 180° rotation, only one rotation is generated by each of the six pairs of opposite edges. These six rotations bring the count to 23.

Then, as promised earlier, we count the identity, **I**, bringing the count up to 24.

15.4 Finding a Matrix

Now that we have enumerated the 24 rotations of the cube, the question naturally arises: since the rotation **R** in §**15.3** was represented by a matrix, can we find matrix representations for *all* rotations of the cube?

Of course the reader must, by now, know the answer to such a leading question. To explain the technique, we consider an example.

Let **M** be the matrix

$$\mathbf{M} = \begin{pmatrix} 4 & 3 & -2 \\ 9 & 5 & 6 \\ -8 & 7 & 12 \end{pmatrix}.$$

Using (15.5), the reader may easily verify that

$$\mathbf{M}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}4\\9\\-8\end{pmatrix},\quad \mathbf{M}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}3\\5\\7\end{pmatrix},\quad \mathbf{M}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\6\\12\end{pmatrix}.$$

But the results of these operations are just the columns of **M**!

So if we reverse this procedure, we arrive at the following: if **R** is a rotation (or any other linear function), the three columns of the matrix reprentations for **R** are given by

$$\mathbf{R}\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \mathbf{R}\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \mathbf{R}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Let us see how this works with an example. Consider the clockwise 90° rotation about the x-axis. That is, your viewpoint is from the positive x-axis, and you're turning the cube a one-quarter rotation to your right. If we call this rotation **R**, what is the matrix for **R**?

We first examine $\mathbf{R}((1,0,0))$. Now $(1,0,0)$ is just the center of the front face of the cube. When we rotate the cube, this point does not move as it is on the axis of rotation. Thus,

$$\mathbf{R}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

What happens to $(0,1,0)$? This point is the center of the right face (looking from our viewpoint). When we rotate, this point is moved to the center of the bottom face of the cube. Thus,

$$\mathbf{R}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\-1\end{pmatrix}.$$

Finally, we look at $(0, 0, 1)$, the center of the top face of the cube. After rotation, this point becomes the center of the right face of the cube, so that

$$\mathbf{R}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Combining our observations, we see that the matrix for $\mathbf{R}$ is given by

$$\mathbf{R}=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}. \tag{15.6}$$

Thus, a matrix representation may be found for *any* symmetry of the cube in the manner just described. We'll continue working on this task in the next few sections.

15.5 Reflections

Not all symmetries of the cube are rotations. Consider for a moment the xy-plane as a mirror in which we may *reflect* the cube. In other words, we reflect the top face to the bottom, and *vice versa,* through the plane $z = 0$. This is easy to describe algebraically as well, for it becomes a simple matter of changing the sign of the z-coordinate. The plane $z = 0$ is called a **plane of symmetry** because it divides the cube into two identical pieces, each a mirror image of the other. Using (15.5), it is easy to see that the matrix which changes the sign of the z-coordinate is given by

$$\mathbf{S}_z=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}. \tag{15.7}$$

Of course, the xz-plane and the yz-plane are also planes of symmetry for analogous reasons. Algebraically, these change the signs of the y-coordinate and x-coordinate, respectively, and have matrix representations

$$\mathbf{S}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{S}_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

Does the cube have other planes of symmetry? (The reader may wish to pause here and consider the question before going further.) Imagine a plane passing through a pair of opposite edges of the cube. Such a plane also divides the cube into two identical pieces, each the mirror image of the other. Since there are twelve edges on the cube in six opposite pairs, this line of thought provides six more planes of symmetry of the cube, and thus six additional reflections.

So the cube has nine planes of symmetry, each generating a reflection. But there are other ways to generate reflections of the cube. For example, select any of the 24 rotations described earlier, and consider the symmetry

$$\mathbf{S}_z\mathbf{R} = \mathbf{S}_z \circ \mathbf{R}.$$

Recall the convention of writing matrix composition as "multiplication," and also recall that for any point p,

$$\mathbf{S}_z \circ \mathbf{R}(p) = \mathbf{S}_z(\mathbf{R}(p)),$$

so that function application is read "right-to-left." This symmetry first rotates the cube as described by $\mathbf{R}$, and then reflects it in the xy-plane as described by $\mathbf{S}_z$. Thus $\mathbf{S}_z\mathbf{R}$ leaves the cube occupying the same region of space as it originally did, and hence is a symmetry of the cube. Since each rotation $\mathbf{R}$ generates a different reflection $\mathbf{S}_z\mathbf{R}$, the 24 reflections of the cube may be produced in this way.

To find the matrix representation for a reflection, recall that it is sufficient to determine the action of the reflection on the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and use the results as the columns for its matrix. Another way to produce the matrix is by using a generalization of (15.5), which is a formula for matrix multiplication:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}x_1 + a_{12}y_1 + a_{13}z_1 & a_{11}x_2 + a_{12}y_2 + a_{13}z_2 & a_{11}x_3 + a_{12}y_3 + a_{13}z_3 \\ a_{21}x_1 + a_{22}y_1 + a_{23}z_1 & a_{21}x_2 + a_{22}y_2 + a_{23}z_2 & a_{21}x_3 + a_{22}y_3 + a_{23}z_3 \\ a_{31}x_1 + a_{32}y_1 + a_{33}z_1 & a_{31}x_2 + a_{32}y_2 + a_{33}z_2 & a_{31}x_3 + a_{32}y_3 + a_{33}z_3 \end{pmatrix}. \tag{15.8}$$

It is readily seen that to multiply matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, take the columns of $\mathbf{M}_2$, find their images under $\mathbf{M}_1$ using (15.5), and use the results as columns for $\mathbf{M}_1\mathbf{M}_2$. The reader should check that if $\mathbf{R}$ is given by (15.6) and $\mathbf{S}_z$ is given by (15.7), the matrix $\mathbf{S}_z\mathbf{R}$ is given by

$$\mathbf{S}_z\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

15.6 Direct and Opposite Symmetries

To date, we have generated 24 rotations of the cube along with 24 reflections of the cube. The set of rotations is sometimes called the **rotation group** of the cube or the **octahedral group** (since it is also the rotation group for the octahedron), while the set including the rotations and reflections is called the **symmetry group** of the cube. The term "group" here means more than simply a collection of objects; the reader interested in group theory should consult an abstract algebra text.

Rotations are called **direct** symmtries, while reflections are called **opposite** symmetries. We just learned how to generate them, but now ask a more subtle question: given a symmetry of cube, such as

$$\mathbf{S} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

how can we determine if it is direct or opposite?

One way is to puzzle out the geometry of the symmetry by looking at its effect on the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. There is also an algebraic method, taken from linear algebra, involving the **determinant** of a matrix, defined by

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \quad (15.9)$$

At first glance, this formula looks rather strange until we remember we are isolating one definition from a broad mathematical topic (linear algebra) as an aid in performing a certain calculation. The interested reader should consult a relevant textbook.

There are many ways to calculate the determinant, each yielding the same result, of course. One popular mnemonic device for calculating determinante of 3 × 3 matrices is as follows. First, copy the first two columns of the matrix to its right, as in Figure 15.1. Take the products of terms along the "up" arrows, add them, then subtract the products of terms along the "down" arrows.

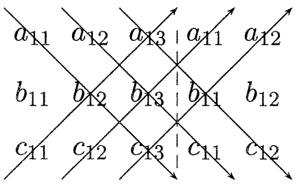

Figure 15.1

Try this with the symmetry **S** given above. It turns out that all products are 0 except one, making the calculation $\det \mathbf{S} = -1$ somewhat easier. And now the reason for introducing the determinant in the first place: if **S** is a rotation, then $\det \mathbf{S} = 1$, while if **S** is a reflection, $\det \mathbf{S} = -1$.

Another nice property of the determinant is that it is multiplicative. In other words, if $\mathbf{M}_1$ and $\mathbf{M}_2$ are two matrices, then

$$\det(\mathbf{M}_1\mathbf{M}_2) = (\det \mathbf{M}_1)(\det \mathbf{M}_2). \quad (15.10)$$

One consequence of this result is that the product of two opposite symmetries is direct. (It would be instructive to take a moment to try and see this now.) To see this, let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two reflections; that is, opposite symmetries. Then using (15.10),

$$\det(\mathbf{S}_1\mathbf{S}_2) = (\det \mathbf{S}_1)(\det \mathbf{S}_2) = (-1)\cdot(-1) = 1.$$

But since $\det(\mathbf{S}_1\mathbf{S}_2) = 1$, then $\mathbf{S}_1\mathbf{S}_2$ must be a rotation (a direct symmetry).

Of course this presupposes that if $\mathbf{S}_1$ and $\mathbf{S}_2$ are reflections, then $\mathbf{S}_1\mathbf{S}_2$ is a symmetry of the cube. But the product of any two symmetries is *always* another symmetry; this is one reason the symmetries form a *group.* If a symmetry leaves a cube occupying the same region of space, then *any* product of symmetries also leaves a cube occupying the same region of space, and hence is also a symmetry of the cube.

15.7 A Final Look

As the reader has undoubtedly surmised, there is much more that can be said about symmetries of the cube. We content ourselves with a summary description of the matrices representing symmetries of the cube.

Note that the matrices considered thus far have the property that in every row and column there are two zeroes, along with a $+1$ or -1. This is true of all symmtries of the cube. Considering matrices exclusively with $+1$ terms, we get the following six, where the first three are direct symmetries and the latter three are opposite symmetries.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

Now we consider changing some of the $+1$ terms to -1. Take the example of the first matrix listed above, the identity matrix. With three nonzero positions, there are 2^3 possible ways (including $\mathbf{I}$ itself) to fill these positions with either $+1$ or -1. Similarly, there are 8 symmetries of the cube associated with each of the other five matrices obtained by taking all possible changes of signs of the 1's in those matrices. Thus, the $8 \times 6 = 48$ symmetries of the cube are produced.

This enumeration is nice because it provides a purely algebraic way of describing the symmetries of the cube. But this is just what the budding computer graphicist needs, in addition to a working knowledge of coordinates for the vertices of polyhedra, in order to enter the sublime world of polyhedral graphics.

15.8 Exercises

1. Decide which of the following functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ are linear. For those which are linear, give a matrix representing the function.
   (a) $\mathbf{A}((x,y,z)) = (1,0,0)$,
   (b) $\mathbf{B}((x,y,z)) = (y,z,x)$,
   (c) $\mathbf{C}((x,y,z)) = (x^2,y^2,z^2)$,
   (d) $\mathbf{D}((x,y,z)) = (3x-y, x+y-z, z-y)$,
   (e) $\mathbf{E}((x,y,z)) = (x-1, y+z, z-3)$,
   (f) $\mathbf{F}((x,y,z)) = (-y,-x,-z)$.

2. Of the functions described in the previous exercise, which are symmetries of the cube? Rotations? Reflections?

3. Perform the following operations involving matrices:
   (a)
$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 0 & 1 \\ -1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix},$$
   (b)
$$\begin{pmatrix} x & 0 & 1 \\ 3 & y & 7 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix},$$
   (c)
$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 5 \\ 1 & -3 & 2 \end{pmatrix},$$

(d)

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 4 & -9 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be given as follows:

$$\mathbf{M}_1 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} 3 & 4 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 0 \end{pmatrix}.$$

Find each of the following:

(a) $\mathbf{M}_1((0,0,0))$,
(b) $\mathbf{M}_2((0,1,0))$,
(c) $\mathbf{M}_1((3,-1,2))$,
(d) $\mathbf{M}_2((4,-5,6))$,
(e) $\mathbf{M_1M_2}$,
(f) $\mathbf{M_2M_1}$,
(g) $\det \mathbf{M}_1$,
(h) $\det \mathbf{M}_2$,
(i) $\det \mathbf{M_1M_2}$,
(j) $\det \mathbf{M_2M_1}$.

5. Describe the effects of the following symmetries on the cube in Figure 14.3.

(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

(b)

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

(c)

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(d)

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. Recall that the octahedron and cube are dual polyhedra (see Chapter 9). With this in mind, write a convincing argument that the symmetry group of the octahedron is the same as the symmetry group of the cube.

7. Find all symmetries of the cube in Figure 14.3 with the property that the center of the front face is taken to the center of the top face.

8. Find all symmetries of the cube in Figure 14.3 with the property that the vertex $(1, 1, 1)$ is taken to the vertex $(1, -1, -1)$.

9. Find all symmetries of the cube in Figure 14.3 with the property that the vertex $(1, 1, 1)$ remains fixed and the vertex $(-1, 1, 1)$ is taken to the vertex $(1, 1, -1)$.

10. A symmetry $\mathbf{T}$ of the cube interchanges the positive x-axis and positive y-axis, and also satisfies $\mathbf{T}((1, 1, 1)) = (1, 1, -1)$. Find a matrix for $\mathbf{T}$.

11. It $\mathbf{M}$ is a matrix, there may be a matrix $\mathbf{N}$ with the property

$$\mathbf{MN} = \mathbf{NM} = \mathbf{I}.$$

In this case, $\mathbf{N}$ is called the **inverse of M** and is denoted by $\mathbf{M}^{-1}$. The matrix $\mathbf{S}_z$ in (15.7) satisfies

$$\mathbf{S}_z \circ \mathbf{S}_z = \mathbf{I},$$

so $\mathbf{S}_z$ is its own inverse; i.e., $\mathbf{S}_z^{-1} = \mathbf{S}_z$. This makes sense, since applying $\mathbf{S}_z$ twice undoes the effect of $\mathbf{S}_z$: $\mathbf{S}_z$ reflects about a plane, then applying $\mathbf{S}_z$ again reflects back.

(a) Find the inverse of the matrix $\mathbf{R}$ given in (15.6).

(b) Find the inverse of the matrix **R** given by

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

12. Consider the tetrahedron with vertices $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$ which is inscribed in the cube in Figure 14.3. A moment's thought reveals that any symmetry of the tetrahedron *must* be a symmetry of the cube. However, a symmetry of the cube need not be a symmetry of the tetrahedron (for example, a 90° rotation about the center of a square face takes the tetrahedron into the other regular tetrahedron inscribable in the cube).

    In the language of abstract algebra, the symmetries of the tetrahedron form a **subgroup** of the symmetries of the cube. Find the symmetries of the tetrahedron. Keeping in mind the discussion in the previous paragraph, give a reason for the number of symmetries of the tetrahedron. How many are direct? How many are opposite?