

Chapter 1

Angles and Constructions

Throughout this chapter, the reader is assumed to be familiar with the usual angles in the plane, basic geometric constructions, and basic trigonometry. A brief review of these ideas is included in Chapter 0 and the appendices.

1.1 Angles

In this Chapter, all angles will be measured in degrees. Other units of measurement may be employed later on; the reader will be given sufficient notice when this occurs.

Table 1.1 lists some commonly used values of trigonometric functions. With formulae such as $\sin(180^\circ - \alpha) = \sin \alpha$, $\cos(180^\circ - \alpha) = -\cos \alpha$, etc., we may easily obtain other values as necessary. For simplicity, we use the abbreviation

$$\tau = (1 + \sqrt{5})/2. \tag{1.1}$$

τ is called the “golden ratio.” We undertake a brief geometric exploration to see why this ratio is of such significance.

Consider the pentagon in Figure 1.1. We leave it to the reader to see that Δpqr and Δqst are similar isosceles triangles with apex angle 36° and base angles of 72° . Call the ratio of the length of the longer sides to that of the shorter side of either triangle ρ . The notation “[pq]” denotes the length of the segment pq ; with $x = [pq] = [qr] = [qt]$ and $y = [pr] = [rs]$, consideration of these similar triangles reveals that

$$\rho = \frac{x}{y} = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{1}{\rho}.$$

Multiplying through by ρ yields $\rho^2 = \rho + 1$ and hence the quadratic equation $\rho^2 - \rho - 1 = 0$. By applying the usual quadratic formula, we see that this equation has one positive and one negative root. Since our ratio is positive, we choose the positive root, $(1 + \sqrt{5})/2$. This number is often referred to by the Greek letter, τ .

It seems that τ is one of those numbers which pops up everywhere in mathematics. In geometry, whenever regular pentagons or decagons are mentioned, you can be sure that τ is lurking nearby.

$\cos 0^\circ = 1,$	$\sin 0^\circ = 0,$
$\cos 18^\circ = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}} = \frac{1}{2}\sqrt{\tau+2},$	$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{1}{2\tau} = \frac{1}{2}(\tau-1),$
$\cos 30^\circ = \frac{\sqrt{3}}{2},$	$\sin 30^\circ = \frac{1}{2},$
$\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \frac{\tau}{2},$	$\sin 36^\circ = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}} = \frac{1}{2}\sqrt{3-\tau},$
$\cos 45^\circ = \frac{1}{\sqrt{2}},$	$\sin 45^\circ = \frac{1}{\sqrt{2}},$
$\cos 54^\circ = \frac{1}{2}\sqrt{3-\tau},$	$\sin 54^\circ = \frac{\tau}{2},$
$\cos 60^\circ = \frac{1}{2},$	$\sin 60^\circ = \frac{\sqrt{3}}{2},$
$\cos 72^\circ = \frac{1}{2}(\tau-1),$	$\sin 72^\circ = \frac{1}{2}\sqrt{\tau+2},$
$\cos 90^\circ = 0,$	$\sin 90^\circ = 1.$

Table 1.1

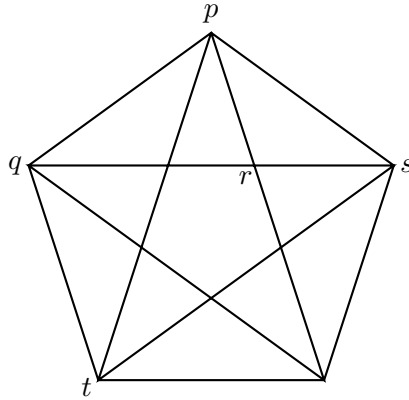


Figure 1.1

1.2 Regular Polygons

As an application of these ideas, we examine a few constructions in the plane. The first is the construction of the regular pentagon (see Figure 1.2). We proceed as follows:

1. Construct a circle with center O .
2. Construct perpendicular lines through O ; label two of the intersections of these lines with the circle a and c (as in Figure 1.2).
3. Construct b as the midpoint of Oa . (A midpoint is usually constructed using a perpendicular bisector, hence the vertical segment through b in Figure 1.2.)
4. Construct d on \overleftrightarrow{Oa} such that $[bd] = [bc]$.
5. $[cd]$ is the length of the side of a regular pentagon inscribed in the circle.

Why does this construction yield a regular pentagon? Consider for a moment a regular pentagon inscribed in a circle of radius 1. Let s denote the length of the sides of the pentagon. Since each edge of the pentagon subtends an angle of 72° with the center of the circle, we may apply the cosine law for triangles, yielding

$$s^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 72^\circ.$$

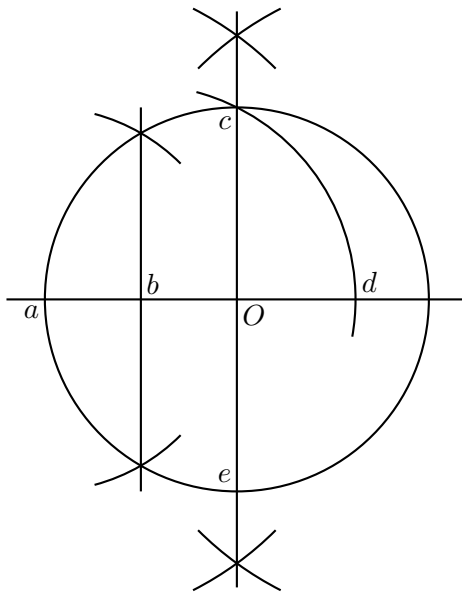


Figure 1.2

Using the value of $\cos 72^\circ$ from Table 1.1, we find that $s^2 = 3 - \tau$, so that $s = \sqrt{3 - \tau}$.

Now we must verify that our construction results in $[cd] = \sqrt{3 - \tau}$, assuming that $[Oa] = [Oc] = 1$. Since $[Ob] = \frac{1}{2}$, an application of the Pythagorean theorem to ΔbOc gives $[bc] = \sqrt{5}/2$. Hence $[bd] = \sqrt{5}/2$, and therefore $[Od] = [bd] - [bO] = (\sqrt{5} - 1)/2$. Another application of the Pythagorean theorem, this time to ΔcOd , yields

$$[cd]^2 = [Oc]^2 + [Od]^2 = 1^2 + \left(\frac{\sqrt{5} - 1}{2}\right)^2 = 1 + \frac{6 - 2\sqrt{5}}{4} = \frac{5 - \sqrt{5}}{2} = 3 - \tau.$$

Thus, $[cd] = \sqrt{3 - \tau}$, and our construction is therefore valid.

The construction of the regular decagon is closely related. In Figure 1.3, we assume that a regular decagon has been inscribed in a circle of radius 1. To find s , the length of the sides of the decagon, we again apply the cosine law for triangles. This yields

$$s^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 36^\circ = 2 - 2 \cdot \frac{\tau}{2} = 2 - \tau,$$

so that $s = \sqrt{2 - \tau}$.

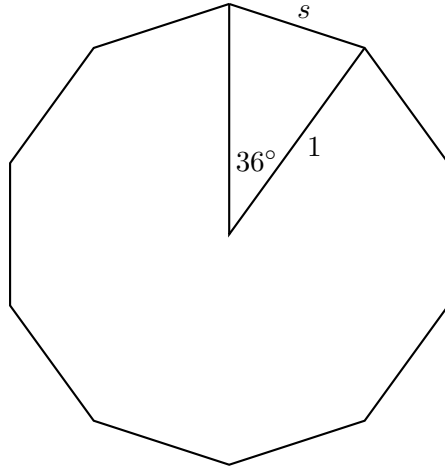


Figure 1.3

What is $\sqrt{2-\tau}$? A short digression on calculations involving τ is in order since such calculations will occur frequently. We first examine powers of τ . Since $\tau^2 = \tau + 1$, we know that

$$\tau^3 = \tau^2 + \tau = (\tau + 1) + \tau = 2\tau + 1.$$

Likewise,

$$\tau^4 = \tau^3 + \tau^2 = (2\tau + 1) + (\tau + 1) = 3\tau + 2.$$

Of course, we may keep going and calculate τ^5 , τ^6 , etc. On the other hand, dividing the relationship $\tau^2 = \tau + 1$ by τ results in $\tau = 1 + 1/\tau$, so that

$$\frac{1}{\tau} = \tau^{-1} = \tau - 1.$$

Proceeding as above, we have $1 = \tau^{-1} + \tau^{-2}$, so that

$$\tau^{-2} = 1 - \tau^{-1} = 1 - (\tau - 1) = 2 - \tau.$$

Again, we may calculate τ^{-3} , τ^{-4} , etc. A short table of values for powers of τ is listed below.

$$\begin{array}{ll}
\tau^1 = \tau, & \tau^{-1} = \tau - 1, \\
\tau^2 = \tau + 1, & \tau^{-2} = 2 - \tau, \\
\tau^3 = 2\tau + 1, & \tau^{-3} = 2\tau - 3, \\
\tau^4 = 3\tau + 2, & \tau^{-4} = 5 - 3\tau, \\
\tau^5 = 5\tau + 3, & \tau^{-5} = 5\tau - 8, \\
\tau^6 = 8\tau + 5, & \tau^{-6} = 13 - 8\tau.
\end{array}$$

Table 1.2

The reader familiar with the famous **Fibonacci sequence** will no doubt find this table intriguing. For those unfamiliar, the Fibonacci sequence is defined as follows: put $F_0 := 0$, $F_1 := 1$, and calculate the remaining members of the sequence by the recurrence relation

$$F_{n+2} := F_{n+1} + F_n$$

when $n \geq 0$. Thus, when $n = 0$, we have $F_2 = F_1 + F_0 = 1 + 0 = 1$; when $n = 1$, we have $F_3 = F_2 + F_1 = 1 + 1 = 2$, etc. This results in the sequence

$$F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, \dots = 0, 1, 1, 2, 3, 5, 8, 13, \dots$$

Of course these are precisely the numbers occurring in Table 1.2. In fact, the whole of this table may be summarized by the following rules, valid for $n \geq 1$:

$$\tau^n = F_n\tau + F_{n-1}, \quad \tau^{-n} = (-1)^{n+1}(F_n\tau - F_{n+1}).$$

Thus, any power of τ , whether positive or negative, may be written in the form $a\tau + b$, where a and b are integers.

Returning to our decagon, we found that $s = \sqrt{2 - \tau}$. But according to our table, $2 - \tau = \tau^{-2}$, so that

$$s = \tau^{-1} = \frac{1}{\tau} = \frac{\sqrt{5} - 1}{2}.$$

We remark here that any expression involving “ τ ” may be converted into an expression involving only rationals and “ $\sqrt{5}$ ” by using $\tau = (\sqrt{5} + 1)/2$ and rationalizing the denominator as necessary. Conversely, expressions involving “ $\sqrt{5}$ ” may be written as expressions involving “ τ ” by using the relationship $\sqrt{5} = 2\tau - 1$.

Now recall that in Figure 1.2, we have $[Od] = (\sqrt{5} - 1)/2$. Thus, in creating our regular pentagon, we have created our decagon as well! This might not have been so obvious if we failed to convert $s = \tau^{-1}$ into an expression involving “ $\sqrt{5}$.”

1.3 Polygon Variations

In addition to the regular polygons, there are a number of other polygons which will be discussed later on. Those we will consider happen to be equiangular, although the lengths of the sides may vary.

—A hexagon variation.

One such polygon is a hexagon, each of whose angles has measure 120° , but whose sides alternate in the ratios τ and 1. Such a hexagon is called a $\tau : 1$ -**hexagon**. As can be seen in Figure 1.4, there are two approaches to constructing such a hexagon from an equilateral triangle – either inscribe the shorter sides of the hexagon in the triangle or inscribe the longer sides. We examine a construction using the latter approach, relegating the former approach to the Exercises.

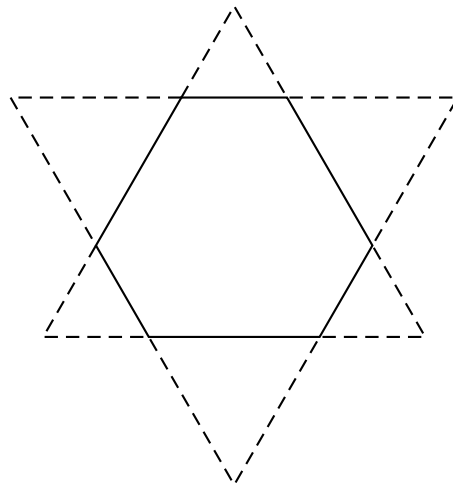


Figure 1.4

This hexagon variation may be constructed as follows (see Figure 1.5). To avoid clutter on the diagram, many of the auxiliary arcs necessary for the construction are omitted.

1. Create an equilateral triangle Δabc .
2. Perpendicularly bisect bc at d .
3. With this same radius $[bd]$, draw an arc centered at d from b counterclockwise to c .
4. With center b and radius $[bd]$, draw an arc counterclockwise from \overrightarrow{ab} .
5. Extend cb past b and draw a perpendicular to this line at b . Denote by g the intersection of this perpendicular with the arc drawn in Step 4.
6. Draw the segment cg to intersect the arc drawn in Step 3 at h .
7. Construct an arc centered at c with radius $[ch]$ to intersect ab at j and k . j and k are vertices of the $\tau : 1$ -hexagon.
8. Draw similar arcs from a and b to complete the construction of the vertices of the hexagon, and then join the vertices to form the sides.

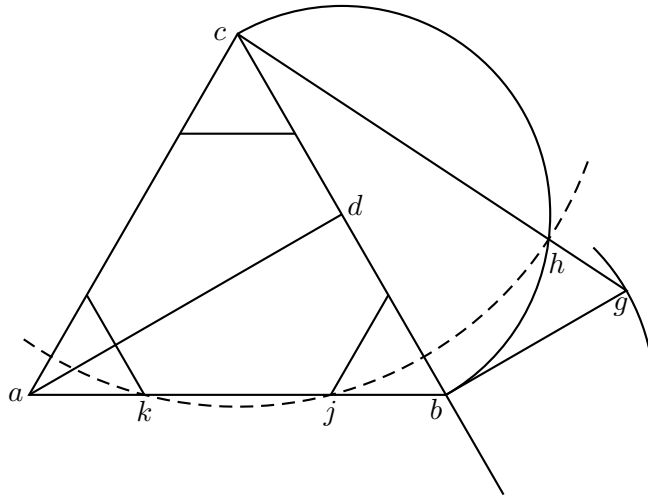


Figure 1.5

Why does this construction yield a $\tau : 1$ -hexagon? To see why, begin with the assumption that $[ab] = [bc] = [ca] = s$. From Step 5, it follows that $\angle cbg$ is a right angle. From Steps 4 and 5, it follows that bd and bg are radii of the same circle, and hence $[bg] = [bd] = \frac{1}{2}s$. From the Pythagorean theorem, we see that $[cg] = \frac{\sqrt{5}}{2}s$.

Now $\angle chb$ is a right angle as it is inscribed in a semicircle. Thus Δcbg and Δchb are similar triangles, giving

$$\frac{[ch]}{[cb]} = \frac{[cb]}{[cg]}.$$

With data from above, we solve for $[ch]$ to see that $[ch] = \frac{2}{\sqrt{5}}s$, and hence from Step 7 that $[cj] = \frac{2}{\sqrt{5}}s$.

We now apply the cosine law for triangles to Δcbj , yielding

$$[cj]^2 = [cb]^2 + [jb]^2 - 2[cb][jb] \cos 60^\circ.$$

Using $[cj] = \frac{2}{\sqrt{5}}s$ and $[cb] = s$, this equation becomes

$$[jb]^2 - [jb]s + \frac{1}{5}s^2 = 0.$$

We wish to establish that $[kj]/[jb] = \tau$, so we use the previous equation with the observation that $s = [kj] + 2[jb]$. This substitution gives

$$[jb]^2 - [jb]([kj] + 2[jb]) + \frac{1}{5}([kj] + 2[jb])^2 = 0,$$

which after some rearrangement becomes

$$[kj]^2 - [kj][jb] - [jb]^2 = 0.$$

Dividing through by $[jb]^2$ yields

$$\left(\frac{[kj]}{[jb]}\right)^2 - \frac{[kj]}{[jb]} - 1 = 0.$$

This implies that

$$\frac{[kj]}{[jb]} = \frac{1 + \sqrt{5}}{2} = \tau \quad \text{or} \quad \frac{[kj]}{[jb]} = \frac{1 - \sqrt{5}}{2} = 1 - \tau.$$

Since the ratio must be positive, we see that $[kj]/[jb] = \tau$, as desired. Hence, our construction does indeed yield a $\tau : 1$ -hexagon.

—**Decagon and decagram variations.**

Happily, it is much easier to construct a $\tau : 1$ -decagon – that is, a decagon all of whose angles have measure 144° but whose sides alternate in the ratios τ and 1. One simply constructs a regular pentagon, trisects the sides, and “connects the dots” (see Figure 1.6(b)). That this yields a $\tau : 1$ -decagon is a consequence of the fact that the dashed lines in this figure are part of 36° – 108° – 36° triangles, and the ratio of the longer to the shorter sides of such triangles is $\tau : 1$.

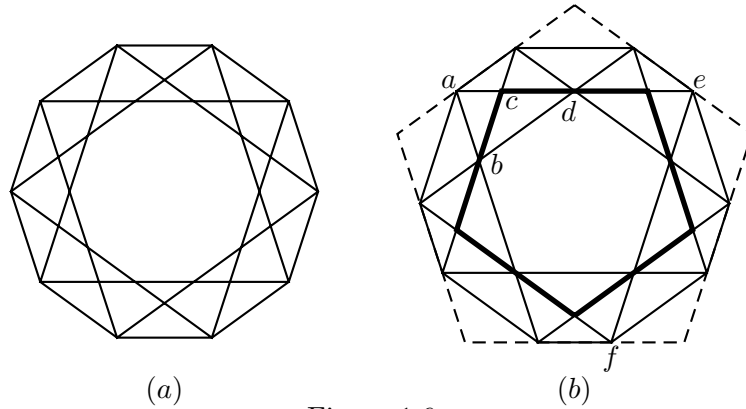


Figure 1.6

By joining every third vertex in a regular decagon, a regular **decagram** is formed (see Figure 1.6(a)). The suffix “-gram” is used when the regular polygon is star-shaped, and hence nonconvex. Similarly, one may construct a decagram by connecting every third vertex of a $\tau : 1$ -decagon (see Figure 1.6(b)). One may show (see Exercise 4) that the sides of this decagram are in the ratio $\sqrt{5} : 2$. By examining the bold pentagon in Figure 1.6(b), one sees an alternative method for constructing a $\sqrt{5} : 2$ -decagram – simply extend the sides of a pentagon and extend the segments joining the midpoints of adjacent sides of the pentagon.

—**Octagon and octagram variations.**

A regular octagon may be formed by dividing the sides of a square in the ratios $1 : \sqrt{2} : 1$ (see Figure 1.7(a)). To obtain an octagon variation, the sides of a square may be divided into the ratios $1 : 1 : 1$ (see Figure 1.7(b)). It is easy to see that the ratio of the length of the longer sides of this octagon to that of the shorter sides is $\sqrt{2} : 1$.

As with the decagon, a regular octagram may be inscribed in a regular octagon by connecting every third vertex (see Figure 1.7(a)). Likewise, an octagram may be inscribed in a $\sqrt{2} : 1$ -octagon by connecting every third

vertex (see Figure 1.7(b)). We see that the sides of the octagram variation come in two different lengths – in Figure 1.7(b), the ratio of the length of the “vertical” sides to the length of the “oblique” sides is $3 : 2\sqrt{2}$, resulting in a $3 : 2\sqrt{2}$ -octagram. A simple construction for creating a $\sqrt{2} : 1$ -octagon is included in the Exercises.

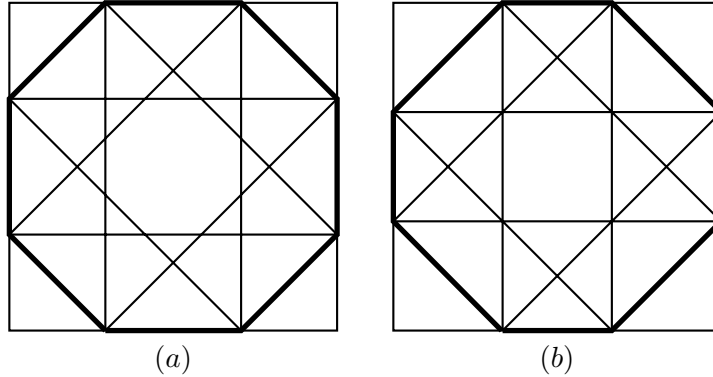


Figure 1.7

Polygons described in this section; that is, equiangular polygons whose sides alternate in length, are called **quasi-regular polygons**.

1.4 Rating a Construction

The following scheme may be used to score a particular construction according to its complexity – every time a procedure with straightedge and compass is done, the corresponding number of points is added to the total. Suggested values are as follows:

1. Drawing an arbitrary line (i.e., not specifying any points on the line) – 0 points.
2. Drawing a line through one specified point (such as drawing a diameter of a circle) – 1 point.
3. Drawing a line through two specified points – 2 points.
4. Extending a given line – 1 point.
5. Opening a compass to a random setting (such as may be needed in a bisection operation) – 0 points.

6. Opening a compass to a specified setting (such as the length of a particular segment) – 2 points.
7. Drawing any arc of a circle with an unspecified center, such as drawing an initial circle in a construction (assuming that the compass is already set to the appropriate radius) – 0 points.
8. Drawing any arc of a circle with a specified center; that is, specifying a point or requiring that the center lie on a given line (assuming the compass is already set to the appropriate radius) – 1 point.

This point system was designed in an attempt to give the simplest, most accurate constructions the lowest values. For example, in setting the compass to the length of a specified segment in the plane, there is some room for human error – and hence this is an “expensive” operation. Drawing an initial line for a construction, however, introduces practically no error – unless your “straightedge” is crooked! Of course, one is free to modify the point system as one sees fit. One might, for example, in order to encourage creativity in using the compass, increase the point value for drawing a line through two points to 10 points.

We illustrate with a sample problem: given a segment of unit length, construct a segment of length $\sqrt{15}$. Let us use the Pythagorean theorem and the fact that any triangle inscribed in a semicircle is a right triangle to consider the following construction (see Figure 1.8):

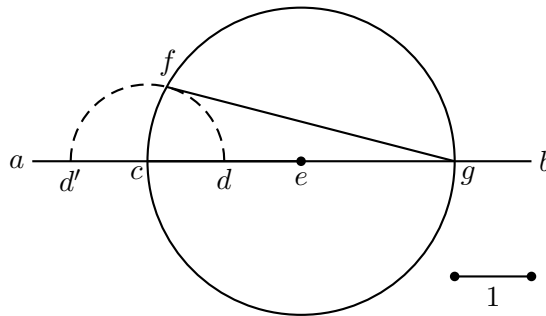


Figure 1.8

1. Set your compass to the unit length – 2 points.
2. Draw an arbitrary line \overleftrightarrow{ab} – 0 points.

3. Choose an arbitrary point c on \overleftrightarrow{ab} , and create a semicircle through d and d' with preset compass – 1 point.
4. With center at d , create e on ab – 1 point.
5. Reset the compass to length $[ce]$ – 2 points.
6. Draw a circle with center e with preset compass – 1 point.
7. With f as the intersection of the semicircle in Step 3 and the circle in Step 6, draw fg – 2 points.

Total for this construction – 9 points.

Since $[cg] = 4$, $[cf] = 1$, and $\angle cfg$ is a right angle (being inscribed in a semicircle), it follows from the Pythagorean theorem that $[fg] = \sqrt{15}$.

1.5 Exercises

1. A regular 15-sided polygon is called a **pentadecagon**, or **15-gon**.
 - (a) Show that the angle subtended by an edge of a pentadecagon from its center is 24° .
 - (b) Using constructions of angles learned in this chapter, construct a 24° angle, and then construct a pentadecagon.
2. Consider the following construction (see Figure 1.9) of a regular pentadecagon:
 - (a) Draw a circle with center O .
 - (b) Draw a horizontal diameter of this circle.
 - (c) Bisect Oa perpendicularly in b (use the compass set to Oa for simplicity).
 - (d) Create a diameter of the circle perpendicular to the one constructed in (b).
 - (e) Create d so that $[bc] = [bd]$ (the pentagon construction again).
 - (f) Bisect Od at e .
 - (g) Create f so that $[ef] = [Od]$.
 - (h) Create g so that $[eg] = [Oa]$.
 - (i) fg is an edge of the 15-gon.

- (b) Create the perpendicular bisector of ac at d .
- (c) Extend ac past a .
- (d) With compass set to $[ad]$, create e on the perpendicular bisector of ac so that $[ad] = [de]$, and f past a so that $[ad] = [af]$.
- (e) With compass set to $[fe]$ and centered at f , create g on ac so that $[fe] = [fg]$.
- (f) With compass set to $[cg]$ and centered at c , create h on cb so that $[cg] = [ch]$.
- (g) With compass still set to $[cg]$, draw an arc centered at b to create j and k , and an arc centered at a to create m and n .

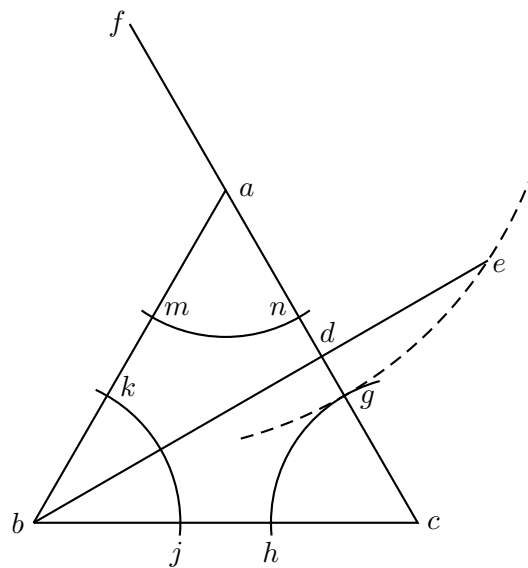


Figure 1.10

Show that $ghjkmn$ is a $\tau : 1$ -hexagon.

4. Recall from §1.1 that the ratio of the length of the longer sides to that of the shorter side of an isosceles triangle with apex angle 36° (such as $\triangle abc$ in Figure 1.6(b)) is τ . Argue geometrically that the ratio of the length of the longer side to that of the shorter sides of an isosceles triangle with apex angle 108° (such as $\triangle bcd$ in Figure 1.6(b)) is also

τ . Use these facts to show that the ratio of the lengths of the sides of the decagram in Figure 1.6(b) is

$$\frac{[ef]}{[ae]} = \frac{\sqrt{5}}{2}.$$

5. Consider the following construction of a $\sqrt{2} : 1$ -octagon. Begin by creating a , b , and c as in the construction of the regular pentagon (see Figure 1.2). Draw a line through c and b , intersecting the circle again at f . Show that af and fe are two sides of a $\sqrt{2} : 1$ -octagon; that is,

$$\frac{[fe]}{[af]} = \sqrt{2}.$$

6. Consider the following construction “system”, given a segment of unit length (see Figure 1.11):

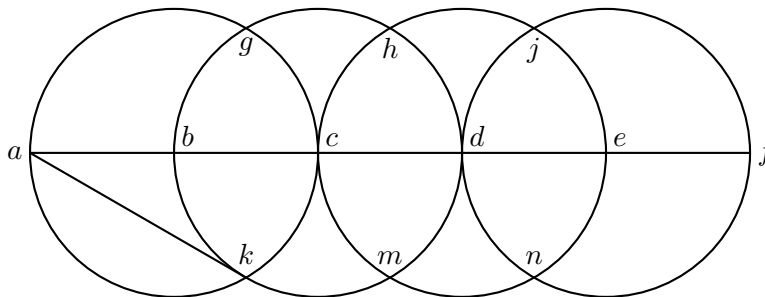


Figure 1.11

- (a) Set your compass to a unit length.
- (b) Draw a circle with the compass at this setting (this circle is the one centered at b in Figure 1.11).
- (c) Draw a (long) horizontal line through the center of this circle (as in Figure 1.11).
- (d) Draw a circle with center c and unit radius.
- (e) Draw a circle with center d and unit radius.
- (f) Draw a circle with center e and unit radius.

Note that this “system” of circles may be extended indefinitely.

- (a) Show that $[gk] = \sqrt{3}$.
 - (b) There are ten segments (that is, segments with endpoints among the points $a-n$) in Figure 1.11 with length $\sqrt{7}$. Find them.
 - (c) With the result of (b) in mind, describe and rate a construction which would yield a segment of length $\sqrt{3+n^2}$, where n is a positive integer.
 - (d) Find a segment in Figure 1.11 of length $\sqrt{13}$.
 - (e) Keeping in mind the result of (d), show that $\sqrt{n^2+n+1}$ (where n is a positive integer) may be produced by a construction whose rating is $n+5$ points. (Hint: Keep constructing more circles until you have enough....)
7. Choose a length at random, say $\sqrt{19}$ or $\sqrt{23}$. Divide into teams, and see who comes up with the most efficient construction using the rating system described in §1.4, or devise a rating system of your own.

