## The Wallis Product

The Wallis Product is a way to compute $\pi$. Discovered by John Wallis in 1655, the Wallis product says that

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}.$$

More precisely,

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots (2n-1)(2n+1)}.$$

In working through this worksheet, you will prove that the Wallis product holds true. Your job is to work through the proof as prescribed, and then to *explain the proof, in essay form, as if your audience did not understand it.* You should try to explain the process of proving the Wallis product as clearly, correctly, and precisely as possible.

Let $I_n = \int_0^{\pi/2} \sin^n(x)\,\mathrm{d}x$ and $J_n = \dfrac{I_{n+1}}{I_n}$. Throughout the worksheet, $n$ represents a non-negative integer.

1. Evaluate $I_0$.

2. Evaluate $I_1$.

3. Use integration by parts to show that

$$\int \sin^{n+2}(x)\,\mathrm{d}x = \frac{-\sin^{n+1}(x)\cos(x)}{n+2} + \frac{n+1}{n+2}\int \sin^n(x)\,\mathrm{d}x.$$

4. Use Question 3 to show that $I_{n+2} = \dfrac{n+1}{n+2}\, I_n$.

5. Explain why it is true that, if $x \in [0, \pi/2]$, then $\sin^{n+2}(x) \le \sin^{n+1}(x) \le \sin^n(x)$.

6. Use Question 5 to show that $I_{n+2} \le I_{n+1} \le I_n$.

7. Divide the inequality from Question 6 by $I_n$ to show that $\dfrac{n+1}{n+2} \le \dfrac{I_{n+1}}{I_n} \le 1$.

8. Use Question 7 to find $\lim_{n\to\infty} J_n$.

9. Evaluate $J_0$.

10. Use Question 4 to show that

$$J_{n+2} = \frac{\left(\dfrac{n+2}{n+3}\, I_{n+1}\right)}{\left(\dfrac{n+1}{n+2}\, I_n\right)} = \frac{(n+2)^2}{(n+1)(n+3)}\, J_n.$$

11. Use Question 10 repreatedly to show that

$$J_{2n} = \frac{(2n)^2}{(2n+1)(2n-1)} \frac{(2n-2)^2}{(2n-1)(2n-3)} \cdots \frac{4^2}{5 \cdot 3} \frac{2^2}{3 \cdot 1} J_0.$$

12. Combine your answers from Questions 8, 9, and 11 to find

$$\lim_{n\to\infty} \frac{(2n)^2}{(2n+1)(2n-1)} \frac{(2n-2)^2}{(2n-1)(2n-3)} \cdots \frac{4^2}{5 \cdot 3} \frac{2^2}{3 \cdot 1}.$$

13. Congratulations! You have just proved the Wallis product! Do you see why?

At this point, clearly you are thinking, "Well, that was awesome, but what does it have to do with Stirling's Formula?" Well, let's find out.

Recall that, in class, we were able to show that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{n} \cdot C$$

for some constant $C$. To find the right value of $C$, we use the Wallis product.

14. The Wallis product says that the limit of the fraction

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots (2n-1)(2n+1)}$$

as $n \to \infty$ is $\pi/2$. Show that this fraction is equal to

$$\frac{(2^n \cdot n!)^2}{\left(\frac{(2n+1)!^2}{(2^n \cdot n!)^2(2n+1)}\right)} = \frac{(2^n \cdot n!)^4(2n+1)}{(2n+1)!^2} \approx \left(\frac{2n}{2n+1}\right)^{4n} e^2C^2 \frac{n^2}{(2n+1)^2}.$$

15. Use L'Hôpital's Rule to compute $\lim_{n\to\infty} \left(\frac{2n}{2n+1}\right)^{4n}$.

16. Use L'Hôpital's Rule to compute $\lim_{n\to\infty} \frac{n^2}{(2n+1)^2}$.

17. Combine your answers from Questions 14-16 to find the limit as $n \to \infty$ of the fraction in the Wallis product in terms of $C$.

18. Solve for $C$. What does this show about Stirling's Formula?