

Extremal Problems for Roman Domination

Erin W. Chambers*, Bill Kinnersley†, Noah Prince‡, Douglas B. West§

Abstract

A *Roman dominating function* of a graph G is a labeling $f: V(G) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The *Roman domination number* $\gamma_R(G)$ of G is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. Let G be a connected n -vertex graph. We prove that $\gamma_R(G) \leq 4n/5$, and we characterize the graphs achieving equality. We obtain sharp upper and lower bounds for $\gamma_R(G) + \gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, improving known results for domination number. We prove that $\gamma_R(G) \leq 8n/11$ when $\delta(G) \geq 2$ and $n \geq 9$, and this is sharp. We conjecture that $\gamma_R(G) \leq \lceil 2n/3 \rceil$ when G is 2-connected, which we prove for a special class.

1 Introduction

According to [3], Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of his cities. He decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself.

The objective, of course, is to minimize the total number of legions needed. The problem generalizes to arbitrary graphs. Define a *Roman dominating function (RDF)* on a graph G to be a vertex labeling $f: V(G) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. For an RDF f , let $V_i(f) = \{v \in V(G) : f(v) = i\}$. In the context of a fixed RDF, we suppress the argument and simply write V_0 , V_1 , and V_2 . Since this partition determines f , we can equivalently write $f = (V_0, V_1, V_2)$. The *weight* $w(f)$ of an RDF f is $\sum_{v \in V(G)} f(v)$, which equals $|V_1| + 2|V_2|$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF of G . Thus, $\gamma_R(G)$ is the minimum number of legions needed to protect cities whose adjacency graph is G .

*Computer Science Department, University of Illinois, Urbana, IL 61801, erinwolf@uiuc.edu. Work supported in part by NSF Graduate Research Fellowship and by NSF grant DMS-0528086.

†Mathematics Department, University of Illinois, Urbana, IL 61801, wkinner2@uiuc.edu.

‡Mathematics Department, University of Illinois, Urbana, IL 61801, nprince@math.uiuc.edu.

§Mathematics Department, University of Illinois, Urbana, IL 61801, west@math.uiuc.edu. Work supported in part by NSA Award No. H98230-06-1-0065.

Roman domination also models other facility location problems. Instead of interpreting $f(v)$ as the number of units placed at v , we can view it as a cost function. Units with cost 2 may be able to serve neighboring locations, while units with cost 1 can serve only their own location. For example, in a communication network, wireless hubs are more expensive but can serve neighboring locations, while wired hubs are low-range but are cheaper.

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [3] began the study of Roman domination, suggested originally in a *Scientific American* article by Ian Stewart [15]. Since $V_1 \cup V_2$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, they observed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G), \tag{1}$$

where $\gamma(G)$ is the domination number of G . In a sense, $2\gamma(G) - \gamma_R(G)$ measures “inefficiency” of domination, since the vertices with weight 1 in an RDF serve only to dominate themselves.

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [3] studied graph theoretic properties of Roman dominating functions and calculated γ_R for specific graphs. They characterized the graphs G such that $\gamma_R(G) \leq \gamma(G) + k$ when $k \leq 2$; this was extended to larger k in [19]. They also characterized graphs G such that $\gamma_R(G) = 2\gamma(G)$ in terms of 2-packings, calling such graphs *Roman*. Henning [9] characterized Roman trees, while Song and Wang [14] characterized the trees T with $\gamma_R(T) = \gamma(T) + 3$. The computational complexity of $\gamma_R(G)$ is considered in [4]. In [12], linear-time algorithms are given for $\gamma_R(G)$ on interval graphs and on cographs, along with a polynomial-time algorithm for AT-free graphs. Other related domination models are studied in [2, 5, 6, 10, 11].

In this paper, we study extremal problems for $\gamma_R(G)$ on various classes of n -vertex graphs. In Section 2, we prove that $\gamma_R(G) \leq 4n/5$ when G is connected and $n \geq 3$, and we determine when equality holds. In Section 3, we obtain sharp upper and lower bounds for $\gamma_R(G) + \gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, where \overline{G} denotes the complement of G . We use these ideas to find the n -vertex graphs G with largest value of $\gamma(G)\gamma(\overline{G})$, shown to equal n in [16].

When G is a connected graph with minimum degree k , inequality (1) and the well-known upper bound on $\gamma(G)$ from [1] and [18] yield $\gamma_R(G) \leq 2\frac{1+\ln(k+1)}{k+1}n$; this was improved slightly in [3], and we use their improvement in Section 3. For small k , the optimal coefficient is of interest. In Section 4, we prove that if G is a connected n -vertex graph with minimum degree at least 2, and $n \geq 9$, then $\gamma_R(G) \leq 8n/11$. The bound is sharp, and we determine when equality holds. In Section 5, we conjecture that if G is also 2-connected, then $\gamma_R(G) \leq \lceil 2n/3 \rceil$, and we prove this bound for a special class.

We use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G , with no loops or multiple edges. The degree of a vertex v in G is $d_G(v)$ or simply $d(v)$. The minimum and maximum vertex degrees are $\delta(G)$ and $\Delta(G)$. For a set $S \subseteq V(G)$, the (*open*) *neighborhood* of S is $\{v \in V(G) - S : v \text{ has a neighbor in } S\}$, denoted $N(S)$. The *closed neighborhood* of S is $N(S) \cup S$, denoted $N[S]$. When $S = \{v\}$, we simply write $N(v)$ and $N[v]$. The *diameter* of G is the maximum distance between vertices of G , denoted $\text{diam } G$. In a tree, a

penultimate vertex is any neighbor of a leaf. We write P_n , C_n , and K_n for the path, cycle, and complete graph with n vertices, respectively. We write mG for the graph consisting of m disjoint copies of G .

2 Connected Graphs

In this section we prove that $\gamma_R(G) \leq 4n/5$ when G is a connected n -vertex graph and characterize when equality holds. Without connectedness, the problem is trivial: always $\gamma_R(G) \leq n$, with equality when $G = \overline{K}_n$.

Since the domination number of a connected n -vertex graph can be as high as $n/2$, (1) only gives $\gamma_R(G) \leq n$, so the bound of $4n/5$ needs additional work. Since deleting an edge cannot decrease γ_R , it suffices to prove the bound for trees.

Theorem 2.1 *If T is an n -vertex tree, with $n \geq 3$, then $\gamma_R(T) \leq 4n/5$.*

Proof. We use induction on n . The base step handles trees with few vertices or small diameter. If $\text{diam } T = 2$, then T has a dominating vertex, and $\gamma_R(T) \leq 2$. This handles $n = 3$. If $\text{diam } T = 3$, then T has a dominating set of size 2, which yields $\gamma_R(T) \leq 4$. This beats the desired bound for such trees with at least six vertices. When one penultimate vertex has degree 2, we put weight 2 on a central vertex and weight 1 on the undominated leaf to stay under the bound.

Hence we may assume that $\text{diam } T \geq 4$. Given a subtree T' with n' vertices, where $n' \geq 3$, the induction hypothesis yields an RDF f' of T' with weight at most $\frac{4}{5}n'$. We find such T' and add a bit more weight to obtain an RDF f of T .

Let P be a longest path in T chosen to maximize the degree of the penultimate vertex v on it, and let u be the non-leaf neighbor of v .

Case 1: $d_T(v) > 2$. Obtain T' by deleting v and its leaf neighbors. Since $\text{diam } T \geq 4$, we have $n' \geq 3$. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = 2$ and $f(x) = 0$ for each leaf x adjacent to v . Since weight 2 on v takes care of its neighbors, f is an RDF for T . Also $w(f) = w(f') + 2 \leq \frac{4}{5}(n-3) + 2 < \frac{4}{5}n$.

Case 2: $d_T(v) = d_T(u) = 2$. Obtain T' by deleting u and v and the leaf neighbor z of v . If $n' = 2$, then T is P_5 and has an RDF of weight 4. Otherwise, the induction hypothesis applies. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = 2$ and $f(u) = f(z) = 0$. Again f is an RDF, and the computation $w(f) < \frac{4}{5}n$ is the same as in Case 1.

Case 3: $d_T(u) > 2$ and every penultimate neighbor of u has degree 2. If every neighbor of u is penultimate or a leaf, then $\text{diam } T = 4$ and T is obtained from a star by subdividing k edges, where $k \geq 2$. Put weight 2 on u and weight 1 on the vertices at distance 2 from u . Now $w(f) = k + 2$ and $n \geq 2k + 1$. With $k \geq 2$, we have $w(f) \leq \frac{4}{5}n$.

Otherwise, some neighbor t of u is neither penultimate nor a leaf. Obtain T' from T by deleting the vertices of the component of $T - tu$ containing u . Now $n' \geq 3$ and the induction

hypothesis applies. Define f on $V(T)$ by $f(x) = f'(x)$ except for $f(u) = 2$, $f(x) = 1$ for each leaf x at distance 2 from u , and $f(x) = 0$ for $x \in N(u) - \{t\}$. Again f is an RDF. We have $w(f) = w(f') + k + 2$, where k is the number of leaves of T at distance 2 from u .

If $k = 1$, then $d_T(u) > 2$ requires at least one leaf neighbor of u . Now $w(f) \leq \frac{4}{5}(n-4)+3 < \frac{4}{5}n$. Otherwise $k \geq 2$, and $w(f) \leq \frac{4}{5}(n - 2k - 1) + (k + 2) = \frac{4}{5}(n + 6 - 3k) \leq \frac{4}{5}n$. \square

As shown in [3], $\gamma_R(P_n) \leq (2n + 2)/3$. The path is not the worst-case n -vertex tree; equality in Theorem 2.1 is achievable. Let L_k consist of the disjoint union of k copies of P_5 plus a path through the central vertices of these copies, as illustrated in Figure 1.

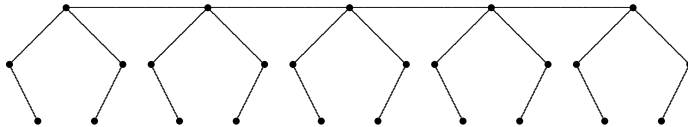


Figure 1: The tree L_5 .

In any RDF of a graph having a leaf with a neighbor of degree 2, an RDF must put weight at least 2 on the pendant edge unless the other neighbor of the 2-valent vertex has weight 2. Thus when two penultimate vertices have a common neighbor, the set of five vertices consisting of the two penultimate vertices, their leaf neighbors, and their common neighbor must receive total weight at least 4. In L_k , there are k disjoint 5-vertex sets of this form, so $\gamma_R(L_k) \geq 4k = 4n/5$. Indeed, we can assemble such copies of P_5 in many ways, and this allows us to characterize the trees achieving equality in Theorem 2.1.

Theorem 2.2 *If T is an n -vertex tree, then $\gamma_R(T) = 4n/5$ if and only if $V(T)$ can be partitioned into sets inducing P_5 such that the subgraph induced by the central vertices of these paths is connected.*

Proof. We have observed that if an induced subgraph H of G is isomorphic to P_5 , and its noncentral vertices have no neighbors outside H in G , then every RDF of G puts weight at least 4 on $V(H)$. Thus in any tree with the structure described, weight at least 4 is needed on every vertex 5-set in the specified partition.

To show that equality requires this structure, we examine the cases more closely in the proof of Theorem 2.1. The proof is by induction on n . In the base cases and Cases 1 and 2, we produce an RDF with weight less than $4n/5$. In Case 3 with diameter 4, equality requires $n = 2k + 1$ and $k = 2$, and the only such tree is P_5 itself.

Define u, T', n', t, k as in the inductive part of Case 3. The bound holds with equality only if $k = 2$ and $n' = n - (2k + 1)$. Thus u has no leaf neighbors, and $T - V(T')$ is a 5-vertex path Q with center u . Equality also requires $\gamma_R(T') = 4n'/5$, so by the induction hypothesis T' has the specified form. In particular, t lies in a copy P' of P_5 in a covering of $V(T')$ by 5-sets inducing paths. Let t' be the center of P' .

If $t \neq t'$, then we build a cheaper RDF for T . Put weight 2 on u and weight 1 on the leaves of Q . Put weight 1 on the neighbor of t in $T' - t'$, and put weight 2 on the penultimate vertex of P' farthest from t . We have now guarded $P' \cup Q$ using total weight 7, and hence $\gamma_R(T) < \frac{4}{5}n$. Hence equality requires $t = t'$ and the specified structure for T . \square

It is easy to extend this characterization to all connected graphs.

Theorem 2.3 *If G is a connected n -vertex graph, then $\gamma_R(G) \leq 4n/5$, with equality if and only if G is C_5 or is the union of $\frac{n}{5}P_5$ with a connected subgraph whose vertex set is the set of centers of the components of $\frac{n}{5}P_5$.*

Proof. If G has the specified form, then as remarked earlier every RDF puts weight at least 4 on the vertex set of each copy of P_5 .

Now suppose that $\gamma_R(G) = \frac{4}{5}n$. Since adding edges cannot increase γ_R , every spanning tree of G has the form specified in Theorem 2.2. Given a spanning tree T , let S_1, \dots, S_k be the 5-sets in the special partition of $V(T)$. The assignment of weight 4 that guards S_i can be chosen independently of any other S_j . If any edge of G joins vertices of S_i and S_j that are not the centers of the paths they induce, then an RDF with weight less than $\frac{4}{5}n$ can be built as in the proof of Theorem 2.2. \square

3 Nordhaus-Gaddum Inequalities

For a graph parameter P , bounds on $P(G) + P(\overline{G})$ and $P(G)P(\overline{G})$ in terms of the number of vertices are called results of ‘‘Nordhaus–Gaddum’’ type, honoring the paper of Nordhaus and Gaddum [13] obtaining such bounds when P is the chromatic number.

For an n -vertex graph G , with $n \geq 2$, it is known (see [8, p. 237]) that

$$3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1 \tag{2}$$

$$2 \leq \gamma(G)\gamma(\overline{G}) \leq n. \tag{3}$$

In this section we determine the analogous sharp results for γ_R .

Proposition 3.1 *If G is an n -vertex graph, then $\gamma_R(G) \leq n - \Delta(G) + 1$.*

Proof. When v is a vertex of maximum degree, the RDF $(N(v), V(G) - N[v], \{v\})$ has weight $n - \Delta(G) + 1$. \square

Theorem 3.2 *If G is an n -vertex graph, with $n \geq 3$, then*

$$5 \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3.$$

Furthermore, equality holds in the upper bound only when G or \overline{G} is C_5 or $\frac{n}{2}K_2$.

Proof. When G has at least three vertices, $\gamma_R(G) \geq 2$, with equality only when G has a dominating vertex. Since a graph and its complement cannot both have dominating vertices, $\gamma_R(G) + \gamma_R(\overline{G}) \geq 5$. Equality holds if and only if G or \overline{G} has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$.

For the upper bound, Proposition 3.1 yields

$$\begin{aligned} \gamma_R(G) + \gamma_R(\overline{G}) &\leq (n - \Delta(G) + 1) + (n - \Delta(\overline{G}) + 1) \\ &= n - \Delta(G) + \delta(G) + 3 \leq n + 3. \end{aligned}$$

Now consider equality. If $\gamma_R(G) + \gamma_R(\overline{G}) = n + 3$, then equality holds throughout the above calculation, and $\delta(G) = \Delta(G)$. Hence G is k -regular for some k . We may assume that $k \leq (n - 1)/2$, since our argument is symmetric in G and \overline{G} . Since equality holds, $\gamma_R(G) = n - k + 1$ and $\gamma_R(\overline{G}) = k + 2$.

Let $v \in V(G)$. If in G some vertex outside $N[v]$ has at least two neighbors outside $N[v]$, then the RDF $(N(u) \cup N(v), V(G) - N[u] - N[v], \{u, v\})$ has weight at most $n - k$, a contradiction. Hence every vertex not in $N[v]$ has at least $k - 1$ neighbors in $N(v)$. A similar argument shows that each vertex in $N(v)$ has at most two neighbors outside $N[v]$.

Letting m be the number of edges joining $N(v)$ and $V(G) - N[v]$, we thus have $(k - 1)(n - k - 1) \leq m \leq 2k$. This simplifies to $n \leq k + 1 + \frac{2}{k-1}$ for $k \geq 2$. Since $n \geq 2k + 1$, we have $k \leq \frac{2}{k-1}$, which requires $k \leq 2$.

For $k = 2$, we have $n \leq k + 1 + \frac{2}{k-1} = 5$, and also $n \geq 2k + 1 = 5$, so the only example is C_5 . For $k = 1$, the only example is $\frac{n}{2}K_2$, where equality holds. For $k = 0$, the only example is $G = \overline{K}_n$, where $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$, and equality does not hold. \square

For the product bound, (1) and (3) yield $\gamma_R(G)\gamma_R(\overline{G}) \leq 4n$. The optimal bound is smaller for sufficiently large n . We will prove in Theorem 3.4 that $\gamma_R(G)\gamma_R(\overline{G}) \leq 16n/5$ when $n \geq 160$. Sharpness is shown by $G = kC_5$, since $\gamma_R(kC_5) = 4k$ and $\gamma_R(\overline{kC_5}) = 4$ and $|V(kC_5)| = 5k$. In fact, equality holds only when G or \overline{G} is kC_5 (when n is large).

The most difficult case in the proof of Theorem 3.4 is when $\text{diam } G = \text{diam } \overline{G} = 2$. We handle this case separately in the next lemma, using a result from Cockayne, Dreyer, Hedetniemi, and Hedetniemi [3]. For an n -vertex graph G , they proved that

$$\gamma_R(G) \leq \frac{2 + 2 \ln((1 + \delta(G))/2)}{1 + \delta(G)} n. \quad (4)$$

Since $\gamma_R(G) \leq 2\gamma(G)$, this bound slightly refines the well-known bound $\gamma(G) \leq \frac{1 + \ln(1 + \delta(G))}{1 + \delta(G)} n$ due to Arnaudov [1] and Payan [18].

Lemma 3.3 *If G is an n -vertex graph, with $n \geq 160$, and $\text{diam } G = \text{diam } \overline{G} = 2$, then $\gamma_R(G)\gamma_R(\overline{G}) < 16n/5$.*

Proof. Let G be such a graph, and let v be a vertex of minimum degree in G . If $d(v) \leq 2$, then the diameter constraint implies that $(V(G) - N(v), \emptyset, N(v))$ is an RDF of G and that $(V(G) - N[v], N(v), v)$ is an RDF of \overline{G} , so $\gamma_R(G)\gamma_R(\overline{G}) \leq 16$. Hence we may assume that $d_G(v) \geq 3$, and similarly $\delta(\overline{G}) \geq 3$.

Let $R = V(G) - N_G[v]$. We select disjoint subsets of $N_G(v)$ dominating R , if any exist, as follows. Initialize $B_0 = N_G(v)$; note that B_0 dominates R , since $\text{diam } G = 2$. If B_i dominates R , then let A_{i+1} be a minimal subset of B_i dominating R , let $B_{i+1} = B_i - A_i$, and increment i . If B_i does not dominate R , then stop, setting $q = i$ and $A^* = B_q$. Thus A_1, \dots, A_q partition $N_G(v) - A^*$, with each A_i being a minimal set that dominates R .

Since A_i is a minimal set that dominates R , there is a vertex $r_i \in R$ having only one neighbor in A_i ; let a_i be this neighbor. Since A^* does not dominate R , there exists $w \in R$ such that $A^* \subseteq N_{\overline{G}}(w)$. Let $S = \{r_1, \dots, r_q\} \cup \{v, w\}$ and $T = \{a_1, \dots, a_q\}$. Now $(V(G) - (S \cup T), T, S)$ is an RDF for \overline{G} , since v dominates R , w dominates A^* , and r_i dominates $A_i - \{a_i\}$. Thus $\gamma_R(\overline{G}) \leq 3q + 4$, which reduces to $3q + 2$ if $A^* = \emptyset$.

Let $U = A_j \cup \{v\}$, where $|A_j| = \min_i |A_i|$. Note that U is a dominating set of G . If $|U| = 2$, then $\gamma_R(G) \leq 4$. Since \overline{G} is connected and $\delta(\overline{G}) \geq 3$, Theorem 2.3 yields $\gamma_R(\overline{G}) < 4n/5$. Hence we may assume that $|U| > 2$, which requires $q \leq \delta(G)/2$.

If $q = 1$, then $\gamma_R(\overline{G}) \leq 7$ and $\gamma_R(G) \leq 2|U| = 2\delta(G) + 2$, so $\gamma_R(G)\gamma_R(\overline{G}) \leq 14(\delta(G) + 2)$. Hence we may assume in this case that $\delta(G) \geq 8n/35 - 2$, but now (4) yields $\gamma_R(G) \leq \frac{2+2\ln(4n/35)}{8/35-1/n}$, and this bound is less than $\frac{16n}{35}$ when $n > 54$, so $\gamma_R(G)\gamma_R(\overline{G}) < 16n/5$.

Hence we may assume that $2 \leq q \leq \delta(G)/2$. Using the RDF $(V(G) - U, \emptyset, U)$ and maximizing over $2 \leq q \leq \delta(G)/2$ (which requires $\delta(G) \geq 4$) yields

$$\gamma_R(G)\gamma_R(\overline{G}) \leq \left(\frac{2\delta(G)}{q} + 2 \right) (3q + 4) = (6\delta(G) + 8) + (6q + \frac{8\delta(G)}{q}) \leq 10\delta(G) + 20. \quad (5)$$

Since $10\delta(G) + 20 < 16n/5$ when $\delta(G) + 2 < 8n/25$, we may assume that $\delta(G) \geq 8n/25 - 2$, and similarly for $\delta(\overline{G})$. By (4), $\max\{\gamma_R(G), \gamma_R(\overline{G})\} \leq \frac{2+2\ln(4n/25)}{8n/25-1}n$. With $n \geq 160$, this bound is less than $16n/95$.

If $q \leq 5$, then $\gamma_R(\overline{G}) \leq 19$. If $q \geq \delta(G)/8$, then $\gamma_R(G) \leq 18$. In these cases we obtain $\gamma_R(G)\gamma_R(\overline{G}) < \frac{16n}{95} \cdot 19 = 16n/5$.

Hence we may assume that $6 \leq q \leq \delta(G)/8$. Now $(2\delta(G)/q + 2)(3q + 4) \leq 22\delta(G)/3 + 44$, which is less than $16n/5$ when $\delta(G) < 24n/55 - 6$, so we may assume that $\delta(G)$ and $\delta(\overline{G})$ are at least $24n/55 - 6$. Now (4) yields

$$\gamma_R(G)\gamma_R(\overline{G}) \leq \left(\frac{(2 + 2\ln(12n/55))n}{24n/55 - 5} \right)^2.$$

The upper bound is less than $16n/5$ when $n \geq 141$. □

This method of proof actually yields $\gamma_R(G)\gamma_R(\overline{G}) = O((n \ln n)^{2/3})$ when $\text{diam } G = \text{diam } \overline{G} = 2$.

Theorem 3.4 *If G is an n -vertex graph, with $n \geq 160$, then*

$$\gamma_R(G)\gamma_R(\overline{G}) \leq \frac{16n}{5},$$

with equality only when G or \overline{G} is $\frac{n}{5}C_5$.

Proof. If G has an isolated vertex or edge, then $\gamma_R(\overline{G}) \leq 3$, and then $\gamma_R(G)\gamma_R(\overline{G}) \leq 3n < 16n/5$. Thus we may assume that each component of G has at least three vertices. Applying Theorem 2.1 to each component now yields $\gamma_R(G) \leq 4n/5$.

If $\text{diam } G \geq 3$, then G has vertices u and v with no common neighbor. Thus $\{u, v\}$ is a dominating set in \overline{G} , and $\gamma_R(\overline{G}) \leq 4$. Then $\gamma_R(G)\gamma_R(\overline{G}) \leq (4n/5)4$ when $\text{diam } G \geq 3$, and similarly when $\text{diam } \overline{G} \geq 3$. Lemma 3.3 produces the desired bound in the remaining case.

Since Lemma 3.3 establishes strict inequality, the only way to achieve equality in this bound is if $\gamma_R(G) = 4n/5$ and $\gamma_R(\overline{G}) = 4$ (or vice versa). If $\gamma_R(\overline{G}) = 4$, then $\delta(G) \geq 2$, so Theorem 2.3 implies that every component of G is a 5-cycle. \square

A similar analysis gives the analogous result for domination number.

Theorem 3.5 *If G is an n -vertex graph, with $n \geq 184$, then equality holds in the bound $\gamma(G)\gamma(\overline{G}) \leq n$ of (3) if and only if $\gamma(G)$ or $\gamma(\overline{G})$ equals n or $n/2$.*

Proof. If G or \overline{G} is K_n , then equality holds.

If $\delta(G) = 1$, then $\gamma(\overline{G}) = 2$, and equality holds if and only if $\gamma(G) = n/2$. It is known (see [8]) that an n -vertex graph G without isolated vertices has domination number $n/2$ if and only if G is C_4 or is obtained from some graph with $n/2$ vertices by adding a pendant edge to each vertex. Thus if $n > 4$ and $\gamma(G) = n/2$, then $\gamma(G)\gamma(\overline{G}) = n$.

For $\delta(G) \geq 2$, McQuaig and Shepherd [17] proved that $\gamma(G) \leq 2n/5$. If also $\text{diam } \overline{G} \geq 3$, then $\gamma(G)\gamma(\overline{G}) \leq 4n/5 < n$. Hence we may assume that both G and \overline{G} have diameter 2.

When $\text{diam } G = \text{diam } \overline{G} = 2$, essentially the same argument (with obvious changes) as in the proof of Lemma 3.3 shows that $\gamma(G)\gamma(\overline{G}) < n$ for $n \geq 184$. We omit the details. \square

4 Minimum Degree 2

In this section, we consider how large γ_R can be for connected n -vertex graphs with minimum degree at least 2. In the n -vertex graph G illustrated in Figure 2, an RDF must give weight 4 to an induced 5-cycle unless one of its vertices has an outside neighbor with weight 2. When there is one such vertex, the remaining copy of P_4 still needs weight 3. Hence each subgraph formed from two 5-cycles and a common neighbor must receive weight at least 8, and we obtain $\gamma_R(G) = 8n/11$.

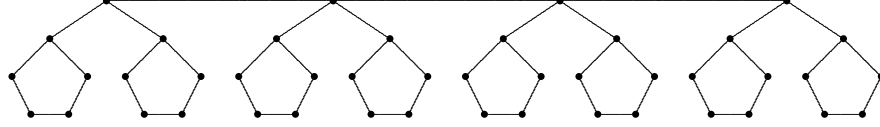


Figure 2: n -vertex graph G with $\gamma_R(G) = 8n/11$.

Lemma 4.1 *Let G be a graph with minimum degree at least 2. If G contains any of the configurations listed below, then there exists a graph G' such that $\delta(G') \geq 2$, $|V(G')| = |V(G)| - 3$, and $\gamma_R(G) \leq \gamma_R(G') + 2$.*

- a) G has an induced path with three internal vertices all having degree 2.
- b) G has an induced 4-cycle C whose vertices alternate between degree 2 and larger degree.
- c) G has an induced 6-cycle C with exactly two vertices of degree at least 3.

Proof. In each case, we define a graph G' with three fewer vertices than G , let f' be an RDF of G' , and produce an RDF f of G with $w(f) \leq w(f') + 2$.

(a) Let u, v, w be the three consecutive vertices of degree 2. Let x and y , respectively, be the neighbors of u and w other than v . Let G' be the graph obtained from G by deleting $\{u, v, w\}$ and adding the edge xy , if x and y are distinct and nonadjacent. Every vertex of G' , including x and y , has the same degree in G' as in G .

Let $f(v) = 2$ and $f(u) = f(w) = 0$, with $f(z) = f'(z)$ for all $z \in V(G) - \{u, v, w\}$. Now f suffices unless $\{f'(x), f'(y)\} = \{2, 0\}$ and the edge xy is needed for f' to be an RDF. By symmetry, we may assume $f'(y) = 0$; let $f(w) = 2$ instead of $f(v) = 2$.

(b) Contract C into a single vertex v to form G' . Let x and y be the vertices of C with degree at least 3 in G ; note that $d_{G'}(v) \geq 2$ and $\delta(G') \geq 2$.

If $f'(v) \in \{1, 2\}$, then let $f(x) = f'(v)$ and $f(y) = 2$, put weight 0 on $V(C) - \{x, y\}$, and let $f(z) = f'(z)$ for $z \in V(G) - V(C)$.

If $f'(v) = 0$, then v has a neighbor a in G' with weight 2. By symmetry, we may assume that $a \in N_G(x)$. Let $f(y) = 2$, put weight 0 on $V(C) - \{y\}$, and let $f(z) = f'(z)$ for $z \in V(G) - V(C)$.

(c) If the vertices x and y with degree at least 3 are not opposite on C , then case (a) applies. Otherwise, form G' by contracting C into a single vertex v and adding a 3-cycle through v and two new vertices. An RDF f' of G' must put total weight at least 2 on the vertices of the 3-cycle. Let $f(x) = f(y) = 2$, put weight 0 on $V(C) - \{x, y\}$, and let $f(z) = f'(z)$ for $z \in V(G) - V(C)$. We have $w(f) \leq w(f') + 2$. \square

Lemma 4.2 *If G is a n -vertex subdivision of a star having a vertex v with $d(v) \geq 3$, then $\gamma_R(G) \leq 8n/11$ unless $d(v) = 3$ and the leaves have distances $(1, 3, 3)$ or $(2, 2, 3)$ from v .*

Proof. Let l_i be the number of leaves at distance i from v . Suppose first that the longest path from v has length at most 3. Note that $n = 1 + l_1 + 2l_2 + 3l_3$.

If $l_1 = l_2 = 0$, then let $f(v) = 1$; there is an RDF with weight $1 + 2l_3$, and $1 + 2l_3 < (24l_3 + 8)/11$ when $l_3 \geq 2$.

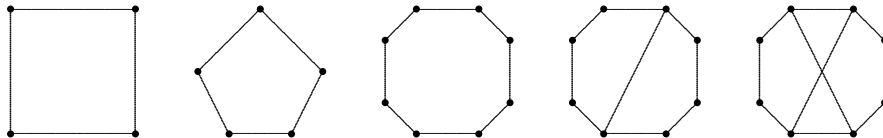
If $l_1 = 0$ and $l_2 = 1$, then put weight 2 on the neighbor of v along the short path. There is an RDF with weight $2 + 2l_3$, and $2 + 2l_3 < (24l_3 + 24)/11$ when $l_3 \geq 0$.

Otherwise, put weight 2 on v to form an RDF with weight $2 + l_2 + 2l_3$. The weight is bounded by the desired value $(24l_3 + 16l_2 + 8l_1 + 8)/11$ when $14 \leq 8l_1 + 5l_2 + 2l_3$. Since we have $l_1 + l_2 + l_3 \geq 3$ and $l_1 + l_2 \geq 1$ with equality only when $l_1 = 1$, the inequality fails only when $(l_1, l_2, l_3) \in \{(1, 0, 2), (0, 2, 1)\}$. In these two cases, $n = 8$ and $\gamma_R(G) = 6$.

With the graphs above as a basis, we now apply induction on n . If some path from v has length more than 3, then let G' be the graph obtained from G by deleting three vertices from the end of the longest path from v . Using weight 2 on the middle of those three vertices yields $w(G) \leq w(G') + 2$. Since $2/3 \leq 8/11$, the induction hypothesis yields $\gamma_R(G) \leq 8n/11$ unless G' is one of the two forbidden 8-vertex graphs above. In this case, $n = 11$ and $\gamma_R(G) \leq 8$, so the desired ratio holds. \square

A *thread* in a graph G is a walk whose internal vertices have degree 2 in G and whose endpoints do not have degree 2. If the endpoints of a thread are equal, then it forms a cycle having one vertex of degree greater than 2. In a connected graph with maximum degree at least 3, the threads decompose the edge set.

Theorem 4.3 *If G is a connected n -vertex graph with $\delta(G) \geq 2$ other than those shown below, then $\gamma_R(G) \leq 8n/11$.*



Proof. Let $\mathcal{C} = \{C_4, C_5, C_8\}$. Note that $\gamma_R(C_4) = 3 > \frac{32}{11}$, $\gamma_R(C_5) = 4 > \frac{40}{11}$, and $\gamma_R(C_8) = 6 > \frac{64}{11}$, and one or two chords added to C_8 as shown above do not help. For $G \in \mathcal{C}$, we have $\gamma_R(G) \leq \frac{8|V(G)|}{11} + \frac{4}{11}$.

For the upper bound, we use induction on n . If G is a cycle, then the claim holds, so we may assume that $\Delta(G) \geq 3$. Our eventual aim is to find a spanning subgraph of G in which one component G_1 is a subdivided star to which we can apply Lemma 4.2, and the remainder G_2 is a graph to which we can apply the induction hypothesis. First we use the induction hypothesis to restrict the structure of G .

Since $2/3 < 8/11$, Lemma 4.1(a) allows us to assume that G has no thread with at least three internal vertices.

Since deleting an edge cannot reduce γ_R , we may assume that every edge joining two vertices with degree at least 3 is a cut-edge. In particular, no cycle in G has a chord. If G has a cut-edge uv with endpoints of degree at least 3, then let H_u and H_v be the components

of $G - uv$ containing u and v , respectively. Both H_u and H_v are edge-minimal connected graphs with minimum degree at least 2. If neither lies in \mathcal{C} , then the RDFs guaranteed for them by the induction hypothesis combine to form the desired RDF of G . If $H_u, H_v \in \mathcal{C}$, then in each case weight 2 on u helps to save one unit on H_v , so

$$\gamma_R(G) \leq \gamma_R(H_u) + \gamma_R(H_v) - 1 \leq \frac{8|V(H_u)| + 4}{11} + \frac{8|V(H_v)| + 4}{11} - 1 < \frac{8n}{11}.$$

Thus when G has a cut-edge uv with $d_G(u), d_G(v) \geq 3$, we may assume that exactly one of $\{H_u, H_v\}$ lies in \mathcal{C} .

Now let v be a vertex of degree at least 3 that is not in a member of \mathcal{C} joined to the rest of G by a cut-edge. By the preceding paragraph, such a vertex exists; at least one end of every thread is suitable. We seek a subgraph G_1 consisting of $d(v)$ paths from v whose lengths do not equal 3, such that $\delta(G - V(G_1)) \geq 2$ and no component of $G - V(G_1)$ lies in \mathcal{C} . By Lemma 4.2 and the induction hypothesis, such a subgraph completes the proof.

Consider the threads emanating from v . If v lies on a cycle C whose other vertices have degree 2, then regardless of the length of C , it is possible to delete one edge e of C so that $C - e$ consists of two threads from v with neither having length 3.

All other threads from v lead to other vertices of degree at least 3. Let u be such a vertex, reached by a thread P . If u has degree 3 and lies on a cycle C whose other vertices have degree 2, then P reaches u via a cut-edge, and $C \in \mathcal{C}$ by our earlier remarks. Cutting an edge e of C incident to u leaves $P \cup C - e$ as a long thread leaving v ; we put it in G_1 .

Otherwise, we want to cut the last edge of P to obtain a short thread leaving v (by Lemma 4.1(a), P has at most two internal vertices). However, cutting two threads that reach u from v could leave u with insufficient degree. By Lemma 4.1(b,c), we may assume that exactly one thread P of length 2 and one thread P' of length 3 reach u from v .

If $d(u) \geq 4$, then we can cut each final edge. If $d(u) = 3$, then a third thread Q leaves u , ending at w . If w is not the end of another thread from v , or if $d(w) \geq 4$, then we can cut the last edge of P or P' (since their lengths differ) so that the resulting thread from v that is formed by cutting the end of Q incident to w does not have length 3.

If w is the end of one other thread from v in G and $d(w) = 3$, then we cut the last edge of P . The thread P' extends to reach w with length at least 3. When we cut the last edge of the other thread from v to w , the thread along P' and Q becomes even longer. The process can continue when v has large degree, yielding one long thread and many short threads.

If the process reaches some w' that is the end of two threads from v , and $d(w') = 3$, then cutting the edge reaching w' leaves a 5-cycle through v whose other vertices have degree 2 (the union of those two threads), and we can cut one edge of it to obtain two short threads.

In the remaining spanning subgraph, the component G_1 containing v is a union of $d(v)$ threads, none having length 3, and every other component has minimum degree at least 2 and is not one of the excluded subgraphs. As remarked above, Lemma 4.2 and the induction hypothesis now provide the desired RDF. \square

To characterize equality in Theorem 4.3, we study its proof closely.

Theorem 4.4 *Let F be the graph of Figure 3. Let G be a connected graph of order n with minimum degree at least 2. If $n \geq 9$, then $\gamma_R(G) = 8n/11$ if and only if*

- (1) $n = 11$ and G is isomorphic to F plus a subset of one of $\{y_1y_3, y_1y_4, y_2y_3, y_2y_4\}$, $\{wz_1, y_1y_3, y_1y_4\}$, or $\{wz_1, wz_3, y_1y_3\}$ added as edges, or
(2) $n > 11$ and G consists of disjoint copies of the graphs F , $F + wz_1$, and $F + wz_1 + wz_2$ with additional edges connecting copies of w .

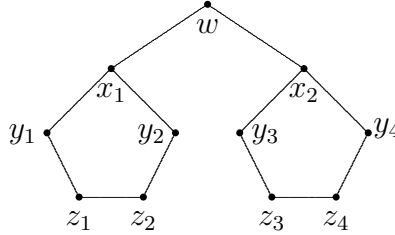


Figure 3: The graph F .

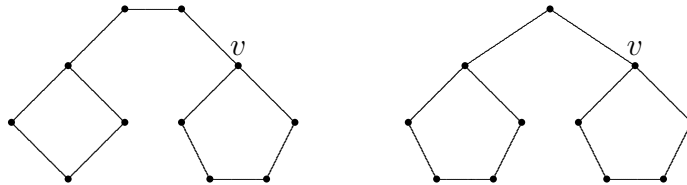
Proof. If G has the indicated form, then, regardless of the edges between copies of w , any RDF must put weight at least 8 on every copy of F , so $\gamma_R(G) \geq 8n/11$.

For the converse, let G be a graph achieving equality in Theorem 4.3. Since $2/3 < 8/11$, G cannot contain a configuration as described in Lemma 4.1. Also the deletion of any cut-edge joining vertices of degree at least 3 without leaving a component in \mathcal{C} must leave components where equality holds.

Let G' be the subgraph resulting from such deletions (called G in Theorem 4.3). Let v be a vertex of G' as chosen in that proof. Since equality holds for G' , it must also hold for the subgraphs G_1 and $G' - V(G_1)$ obtained in the inductive proof.

A closer look at Lemma 4.2 characterizes the vectors of path lengths where $\gamma_R(G_1) = 8|V(G_1)|/11$ can hold. Since the proof of Theorem 4.3 extracts a graph G_1 in which no thread from v has length 3, equality requires the threads from v to have lengths 2, 2, and 6.

To obtain a thread of length 6 without obtaining a thread of length 1, we must have had $d(v) = 3$, and one thread from v reaches a cycle in \mathcal{C} . If $n = 11$, then the possibilities are as shown below, but the graph on the left has an RDF of weight 7. Inspection shows that the only graphs with Roman domination number 8 spanned by F are those claimed.



When $n > 11$, we claim that the endpoints of the threads of length 2 from v are still adjacent and have degree 2. If not, then they would have degree at least 3, and using one

of them in place of v would yield a tree for G_1 that has a thread of length 1 (by cutting the edge of the thread to v). We would then have $\gamma_R(G') < 8n/11$.

We conclude that successively deleting edges of G with endpoints of degree at least 3, without introducing components in \mathcal{C} , yields a graph whose components are copies of F . Since there exist minimum weight RDFs of F putting weight 2 on any given vertex, and deletion of any vertex of F other than w leaves a subgraph where weight 7 suffices, every edge of G not contained among the vertices of a single copy of F joins copies of the central vertex of F .

If any edge of G connects the two 5-cycles in one copy F' of F , then since G is connected, the central vertex w' of F' has a neighbor in another copy of F that can be given weight 2. With w' protected, we can protect the rest of F' with weight 7 using the edge joining the two 5-cycles. This yields $\gamma_R(G) \leq 7 + 8(n - 11)/11 < 8n/11$. Hence no edges can be added between or within the copies of F other than those described in the statement. \square

5 2-Connected Graphs

In this section, we ask whether a connectivity threshold improves the upper bound on Roman domination number. As remarked in the Introduction, connectedness and minimum degree k force $\gamma_R(G) \leq c_k n$, where $c_k = O(\ln k/k)$. Perhaps requiring k -connectedness instead of minimum degree k will reduce the constant in the upper bound or even yield $\gamma_R(G) \leq c'_k n$ with $c'_k = o(c_k)$ as $k \rightarrow \infty$.

Conjecture 5.1 *If G is a 2-connected graph of order n , then $\gamma_R(G) \leq \lceil 2n/3 \rceil$.*

This bound is sharp if true, since $\gamma_R(C_n) = \lceil 2n/3 \rceil$ [3]. Conjecture 5.1 is equivalent to the same statement for *minimally 2-connected graphs*, which are the 2-connected graphs having no edge whose deletion leaves a 2-connected graph. We study minimally 2-connected graphs, using the following characterization.

Definition 5.2 *A Level 0 graph is a cycle of length at least 4. Two vertices x and y in a graph G are compatible in G if no x, y -path in G has a chord. A Level k graph is a graph obtained from graphs G_1, \dots, G_m (with $m \geq 2$), where each G_i an edge or a Level j graph for some $j < k$, in the following way. In each G_i a compatible pair $\{x_i, y_i\}$ is selected, and the graphs are combined by identifying y_i with $x_{i+1} \pmod{m}$, for each i . If $m = 2$, then we do not allow G_1 or G_2 to be a single edge.*

Theorem 5.3 (Dirac [7]) *A graph is minimally 2-connected if and only if it is C_3 or is a Level k graph for some $k \geq 0$.*

We have remarked that Conjecture 5.1 holds for cycles, which include C_3 and all Level 0 graphs; we will prove it for Level 1 graphs.

Theorem 5.4 *If G contains a spanning Level 0 or Level 1 graph, then $\gamma_R(G) \leq \lceil 2n/3 \rceil$.*

Proof. Let $n = |V(G)|$; we use induction on n . In the basis step we include all graphs having spanning cycles. For the induction step, we may assume that G is a Level 1 graph. We may also assume by Lemma 4.1(a) that G has no thread with at least three internal vertices, since contraction of an edge incident to a vertex of degree 2 preserves not only minimum degree 2 but also 2-connectedness.

Let G_1, \dots, G_m be the graphs used in the construction of G according to Definition 5.2. Each G_i is an edge or is a cycle of length at least four. In the latter case, x_i and y_i are nonadjacent. By the exclusion of long threads, when G_i is a cycle the paths joining x_i and y_i on G_i have length 2 or 3, and also at most three consecutive subgraphs in the list are single edges. (If only one G_i is a cycle, then the contraction can produce a cycle with one chord, which is not minimally 2-connected but has a spanning Level 0 subgraph.)

Hence we may assume that each G_i is an edge or is a k -cycle with $k \in \{4, 5, 6\}$. Let $S = \{i: G_i \text{ is a cycle}\}$. For $i \in S$, let i' be the first index in the list $i + 1, i + 2, \dots$ (modulo m) such that $i' \in S$. Let $l = i' - i \pmod m$. We have restricted l to $\{0, 1, 2, 3\}$. For $i \in S$, let $U_i = \left(\bigcup_{j=i}^{i'} V(G_j)\right) - \{x_{i'}\}$. The sets U_i for $i \in S$ partition $V(G)$.

It suffices to specify an RDF f for G that puts weight at most $2|U_i|/3$ on each U_i for $i \in S$. First, let $f(x_i) = 2$ for all $i \in S$. Given $i \in S$, let k be the length of the cycle G_i , and define l as above; we have $|U_i| = k + l - 1$. We must still guard the subgraph H_i induced by $U_i - (N(x_i) \cup N(x_{i'}))$. Let w_i be the total weight we will place on U_i .

If $l = 0$, then H_i is empty and $w_i = 2$, which is small enough since $|U_i| = k - 1 \geq 3$. If $l = 1$, then $H_i = \overline{K}_{k-4}$; put weight 1 on each vertex of H_i . Now $w_i = k - 2$, which is small enough since $|U_i| = k \leq 6$. If $l = 2$, then $H_i = P_{k-3}$ (with center y_i); add weight $\lceil 2(k-3)/3 \rceil$ on y_i to guard H_i . Now $w_i = \lceil 2k/3 \rceil$ and $|U_i| = k + 1$, so w_i is small enough. Finally, if $l = 3$, then $H_i = K_{1,k-3}$ (with center y_i); put weight 2 on y_i to guard H_i . Now $w_i = 4$, which is small enough since $|U_i| = k + 2 \geq 6$. \square

References

- [1] V. I. Arnautov. *Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices*. Prikl. Mat. i Programirovanie 11 (1974) 3–8 (in Russian).
- [2] A. P. Burger, E. J. Cockayne, W. R. Gründlingh, C. M. Mynhardt, J. H. van Vuuren, and W. Winterbach. *Finite order domination in graphs*. J. Combin. Math. Combin. Comput. 49 (2004), 159–175.
- [3] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi. *Roman domination in graphs*. Disc. Math. 278 (2004) 11–22.

- [4] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, S. T. Hedetniemi, and A. A. McRae. *The algorithmic complexity of Roman domination*, submitted.
- [5] E. J. Cockayne, O. Favaron, and C. M. Mynhardt. *Secure domination, weak Roman domination and forbidden subgraphs*. Bull. Inst. Combin. Appl. 39 (2003), 87–100.
- [6] E. J. Cockayne, P. J. P. Grobler, W. R. Gründlingh, J. Munganga, and J. H. van Vuuren. *Protection of a graph*. Util. Math. 67 (2005), 19–32.
- [7] G. A. Dirac. *Minimally 2-connected graphs*. J. Reine Angew. Math. 228 1967 204–216.
- [8] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, 1998: New York.
- [9] M. A. Henning. *A characterization of Roman trees*. Discuss. Math. Graph Theory 22 (2) (2002) 325–334.
- [10] M. A. Henning. *Defending the Roman Empire from multiple attacks*. Discrete Math. 271 (2003), no. 1-3, 101–115.
- [11] M. A. Henning and S. T. Hedetniemi. *Defending the Roman Empire—a new strategy*. The 18th British Combinatorial Conference (Brighton, 2001). Discrete Math. 266 (2003), no. 1-3, 239–251.
- [12] M. Liedloff, T. Kloks, J. Liu, and S.-L. Peng. *Roman domination over some graph classes*. Graph-theoretic concepts in computer science, 103–114, Lecture Notes in Comput. Sci., 3787, Springer, Berlin, 2005.
- [13] E. A. Nordhaus and J. W. Gaddum. *On complementary graphs*. Amer. Math. Monthly 63 (1956), 175–177.
- [14] X. Song and X. Wang. *Roman domination number and domination number of a tree*. Chinese Quart. J. Math. 21 (2006), no. 3, 358–367.
- [15] I. Stewart. *Defend the Roman Empire!* Sci. Amer. 281 (6) (1999) 136–139.
- [16] F. Jaeger and C. Payan. *Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple*. C. R. Acad. Sci., Paris, Ser. A 274 (1972) 728–730.
- [17] W. McQuaig and B. Shepherd. *Domination in graphs with minimum degree two*. J. Graph Theory 13 (6) (1989) 749–762.
- [18] C. Payan. *Sur le nombre d'absorption d'un graphe simple*, in *Colloque sur la Théorie des Graphes (Paris, 1974)*. Cahiers Centre Études Recherche Opér. 17 (1975), 307–317.
- [19] H.-M. Xing, X. Chen, and X.-G. Chen. *A note on Roman domination in graphs*. Discrete Math. 306 (2006), no. 24, 3338–3340.