Minimum Difference Representations of Graphs

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Abstract

Define a k-minimum-difference-representation (k-MDR) of a graph G to be a family of sets $\{S(v) : v \in V(G)\}$ such that u and v are adjacent in G if and only if $\min\{|S(u) - S(v)|, |S(v) - S(u)|\} \geq k$. Define $\rho_{\min}(G)$ to be the smallest k for which G has a k-MDR. In this note, we show that $\{\rho_{\min}(G)\}$ is unbounded. In particular, we prove that for every k there is an n_0 such that for $n > n_0$ 'almost all' graphs of order n satisfy $\rho_{\min}(G) > k$. As our main tool, we prove a Ramsey-type result on traces of hypergraphs.

1 Introduction

A representation of a graph is an assignment of sets of a given type (intervals, convex sets, subsets of $\mathbb{N}$, etc.) to the vertices of G such that $u, v \in V(G)$ are adjacent if and only if $S(u)$ and $S(v)$ satisfy a given condition (are disjoint, intersect, etc.). Many different kinds of graph representations have been studied extensively (see, for example, [5, 8, 10]).

In [7], three new definitions of graph representations were introduced. With each vertex v of G, a finite set $S(v)$ is associated. In the first model, u and v are adjacent if and only if

$$\left(|S(u) - S(v)| + |S(v) - S(u)|\right)/2 \geq k,$$

in the second, iff

$$\max\{|S(u) - S(v)|, |S(v) - S(u)|\} \geq k,$$

and in the third, iff

$$\min\{|S(u) - S(v)|, |S(v) - S(u)|\} \geq k.$$

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Let $\rho_{\mathrm{avg}}(G)$, $\rho_{\max}(G)$, and $\rho_{\min}(G)$ be the smallest k for which G has such a representation in the first, second, and third model, respectively. In [7] it was shown that each of these parameters exists and is at most $n-1$, where n is the order of the graph G. It was also shown that the first two parameters could be large, namely, at least $O(\log n)$. Füredi [9] showed that, for almost all n-vertex graphs, all three parameters are at most $O(n/\log n)$ and $\rho_{\max}(G) = \Omega(n/\log n)$. However, no non-trivial lower bounds on $\rho_{\min}(G)$ have been proven, and in [7] it was noted that no graphs were known for which $\rho_{\min}(G) > 2$.

In this short paper, we will show that $\{\rho_{\min}(G)\}$ is unbounded by demonstrating that, for any $k \in \mathbb{N}$, almost every 'large' graph G satisfies $\rho_{\min}(G) > k$.

Theorem 1 *Let k be a positive integer. Then $G \in \mathbb{G}_{n,1/2}$ satisfies $\rho_{\min}(G) > k$ with high probability as $n \to \infty$.*

Our method is as follows. First, we establish a Ramsey-type result on traces of hypergraphs (Theorem 3) saying that a large enough family of sets must contain one of four configurations. When we apply this to the sets of an optimal k-minimum-difference-representation (k-MDR) of G, that is, $\{S(v) : v \in V(G)\}$, we can immediately rule out three of these (see Lemma 6). The remainder of the proof is dedicated to ruling out the fourth configuration.

Before we begin, let us recall some definitions and notation. A *set system* is a family of finite sets. For two sets A and B, $A \subseteq B$ means A is a subset of B, and $A \subset B$ means that A is a proper subset of B. A set system $\mathcal{H}$ is an *antichain* if no $A, B \in \mathcal{H}$ exist with $A \subset B$. If G is a graph and $X \subseteq V(G)$, then $N_G(X)$ is the set of vertices in $V - X$ which have a neighbor in X. We use $\mathbb{G}_{n,1/2}$ to denote the probability space of all $2^{\binom{n}{2}}$ graphs on n labelled vertices chosen with uniform distribution. Throughout the paper, all logarithms will have base 2.

2 Traces

In this section we prove a Ramsey-type theorem on set systems. Define a *trace* of a set system $\mathcal{H}$ on X to be the system $\mathcal{H}_X = \{E \cap X : E \in \mathcal{H}\}$. We say that $\mathcal{H}_X$ is *induced* by X, and the *trace sets* are the sets $E \cap X$. We say E is an *ancestor in* $\mathcal{H}$ of $E \cap X$. We also consider any subsystem of $\mathcal{H}_X$ to be a trace of $\mathcal{H}$ induced by X.

Define an (ℓ, k)-*star* to be a set system $\{A_i : 1 \le i \le \ell\}$ whose elements are pairwise disjoint and all have cardinality k. An (ℓ, k)-*costar* is a set system $\{A_i : 1 \le i \le \ell\}$ where, if $A = \bigcup_i A_i$, then $\{A - A_i : 1 \le i \le \ell\}$ is an (ℓ, k)-star. An ℓ-star or ℓ-costar is simply a $(\ell, 1)$-star or $(\ell, 1)$-costar, respectively. An m-*double-chain* is a system $\{\{v_i, \ldots, v_{i+m-1}\} : 1 \le i \le m\}$.

Balogh and Bollobás [2] proved the following result on traces of set systems.

Theorem 2 *Let ℓ and m be positive integers and let $g(\ell, m) = (2m)^{\ell} 2^{9^{\ell}}$. Then any antichain $\mathcal{H}$ with $|\mathcal{H}| \ge g(\ell, m)$ contains either an ℓ-star, an ℓ-costar, or an m-double-chain as a trace.*

A short proof of Theorem 2, along with other similar results, appeared in [6]. Recently, this result has seen many applications, for example in [1], [3] and [4].

Our new theorem involves a fourth kind of set system. Define an (r, p, q)-*system* to be a set system

$$\{K \cup F_i \cup \bigcup_{j \neq i} G_j \cup \bigcup_{j \leq i} L_j : 1 \leq i \leq r\}$$

such that

(1) K is the *kernel* of the system,
(2) K and the $3r$ sets F_i, G_i, and L_i for $1 \leq i \leq r$ are pairwise disjoint,
(3) $|F_i| = p$ and $|G_i| = q$ for all $1 \leq i \leq r$.

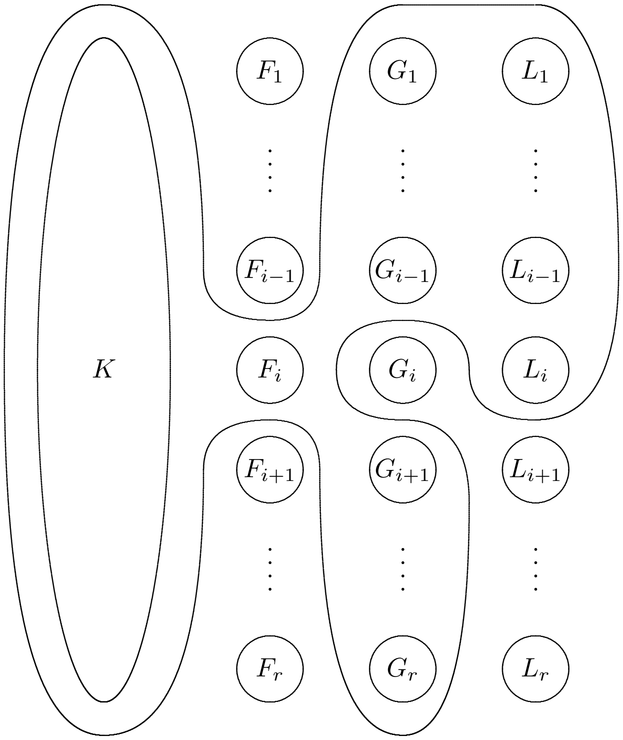

Figure 1. A general element of an (r, p, q)-system

Define $h_1(\ell, m, r) = g(\ell, m)$ and $h_k(\ell, m, r) = g(r^2 \cdot h_{k-1}(\ell, m, r), m)$ for $k > 1$.

Theorem 3 *Let ℓ, k, m, and r be positive integers. Then every antichain $\mathcal{H}$ with $|\mathcal{H}| \geq h_{2k}(\ell, m, r)$ contains an (ℓ, k)-star, an (ℓ, k)-costar, or an m-double-*

chain as a trace or contains an (r,p,q)*-system with* $p,q \leq k$ *as a subsystem. Furthermore, we may insist that the* (r,p,q)*-system have* $L_i = \emptyset$ *for all* i *or* $L_1 = \emptyset$ *and* $L_i \neq \emptyset$ *for all* $i \geq 2$*.*

Throughout the proof, we use the term *collection* to mean a set whose elements may occur with multiplicity greater than 1.

Proof. If $\mathcal{H}$ has an m-double-chain as a trace, we are done. Otherwise, implement the following algorithm:

(Step 1) Initialize $p = q = 0$, $i = 1$ and $\mathcal{H}_1 = \mathcal{H}$.

(Step 2) Let $d_i = r^2 \cdot h_{2k-i}(\ell, m, r)$. If $\mathcal{H}_i$ contains no d_i-star trace, move to Step 3. Otherwise, let $T_1^i, \ldots, T_{d_i}^i$ be the trace sets of a d_i-star trace, increment p by 1, and call the set inducing the star B_p. Let A_j^i be an ancestor in $\mathcal{H}_i$ of T_j^i, and define $\mathcal{H}_{i+1}$ to be the set system $\{A_j^i - B_p : 1 \leq j \leq d_i\}$. Increment i by 1. If $p = k$, stop and output the ancestors in $\mathcal{H}$ of the T_j^i's; these sets contain an (ℓ, k)-star trace. Otherwise repeat Step 2.

(Step 3) If $\mathcal{H}_i$ contains no d_i-costar trace, move to Step 4. Otherwise, let $T_1^i, \ldots, T_{d_i}^i$ be the trace sets of a d_i-costar trace, increment q by 1, and call the set inducing the costar C_q. Let A_j^i be an ancestor in $\mathcal{H}_i$ of T_j^i, and define $\mathcal{H}_{i+1} = \{A_j^i - C_q : 1 \leq j \leq d_i\}$. Increment i by 1. If $q = k$, stop and output the ancestors in $\mathcal{H}$ of the T_j^i's; these sets contain an (ℓ, k)-costar trace. Otherwise return to Step 2.

(Step 4) Find $D_1, D_2, \ldots, D_r$ among the sets $T_1^i, \ldots, T_{d_i}^i$ such that $D_1 = D_2 = \ldots = D_r$ or $D_1 \subset D_2 \subset \ldots \subset D_r$, and let $\mathcal{D} \subset \mathcal{H}$ be the collection of ancestors in $\mathcal{H}$ of D_i's. Stop and output $\mathcal{D}$; these sets contain an (r,p,q)-system with $p, q \leq k$.

We claim this algorithm outputs an (ℓ, k)-star, an (ℓ, k)-costar, or an (r,p,q)-system with $p, q \leq k$.

First, we show the algorithm is well-defined through $2k - 1$ steps. Since $i \leq 2k - 1$, Steps 2 and 3 are well-defined. Therefore, the only thing that must be checked is that when we enter Step 4, we can always find r sets all equal or ordered linearly by proper inclusion. Suppose that we enter Step 4 on the ith turn. If any set appears with multiplicity r among the T_j^i's, then we can take $\mathcal{D}$ to be r distinct ancestors in $\mathcal{H}$ of this set. Otherwise, there are at least $d_i/r = r \cdot h_{2k-i}(\ell, m, r)$ distinct sets among the T_j^i's, so by Dilworth's Theorem there is a chain of length r or an antichain of size $h_{2k-i}(\ell, m, r)$ among the T_j^i's. In the first case we are clearly done. In the second, this antichain contains a d_i-star, a d_i-costar, or an m-double-chain as a trace by Theorem 2. The first two cases are impossible since we did not complete Steps 2 and 3, and the third is impossible by assumption. Hence such $D_1, D_2, \ldots, D_r$ exist.

Second, the algorithm always terminates with $i \leq 2k-1$ since $p+q$ increases by 1 after every complete implementation of Step 2 or 3 and we either stop when $\max(p,q)=k$, or we stop after the first implementation of Step 4.

Finally, we show the algorithm produces what we want. If the algorithm terminates during the ith turn because $p=k$, then we found a star trace $T_1^i, \ldots, T_{d_i}^i$ where $d_i \geq \ell$. Note that the $T_j^i \cap B_n$'s are disjoint and have cardinality 1 for all $1 \leq j \leq d_i$ and $1 \leq n \leq p = k$. Letting A_j^i be an ancestor in $\mathcal{H}$ of T_j^i, we see $A_1^i, \ldots, A_\ell^i$ form an (ℓ, k)-star induced by

$$\left(\bigcup_{n=1}^{p} B_n\right) \cap \left(\bigcup_{i=1}^{\ell} A_j^i\right).$$

The case when the algorithm terminates because $q = k$ is similar. Now, if the algorithm terminated after an implementation of Step 4, let $A_1, \ldots, A_r$ be distinct such that A_i is an ancestor in $\mathcal{H}$ of D_i for $1 \leq i \leq r$. Then the sets $A_i \cap B_n$ and $C_s - A_i$ are pairwise disjoint and have cardinality 1 for all $1 \leq i \leq r$, $1 \leq n \leq p$, and $1 \leq s \leq q$. Let $F_i = \bigcup_{n=1}^{p}(A_i \cap B_n)$, $G_i = \bigcup_{s=1}^{q}(C_s - A_i)$, $L_1 = \emptyset$ and $L_i = D_i - D_{i-1}$ for $2 \leq i \leq r$, and let $K = D_1 \cup (\bigcup_{n=1}^{q} C_n - \bigcup_{i=1}^{r} A_i)$. Then we have, for all i, that

$$A_i = K \cup F_i \cup \bigcup_{j \neq i} G_j \cup \bigcup_{j \leq i} L_j,$$

and $0 \leq |F_i| = p \leq k$ and $0 \leq |G_i| = q \leq k$, so the A_i's form an (r,p,q)-system with kernel K. Note that if the D_i's were all the same, then $L_i = \emptyset$ for all i, whereas if they were all different, $L_1 = \emptyset$ and $L_i \neq \emptyset$ for all $i \geq 2$. □

3 Proof of the main result

In this section, we prove our main result, namely, Theorem 1. Before we begin, we must recall two basic facts about random graphs.

Fact 4 *Almost every* $G \in \mathbb{G}_{n,1/2}$ *has an independent set on* $\log n$ *vertices.*

Fact 5 *Let* $G \in \mathbb{G}_{n,1/2}$ *and let* $S \subseteq V(G)$ *with* $|S| = o(\log n)$. *If* S *is the disjoint union of* A *and* B, *then with high probability there is a vertex* v *such that* v *is adjacent to every vertex in* A *and to no vertex in* B.

We begin with an easy observation.

Lemma 6 *Let* $\mathcal{H}$ *be a set system satisfying* $\min\{|S_i - S_j|, |S_j - S_i|\} < k$ *for all* $S_i, S_j \in \mathcal{H}$. *Then* $\mathcal{H}$ *has no* $(2,k)$*-star,* $(2,k)$*-costar, or* $(k+1)$*-double-chain as a trace.*

Proof. Suppose X induces a $(2,k)$-star trace with trace sets T_1 and T_2. Let S_i be an ancestor in $\mathcal{H}$ of T_i. Then

$$|S_i - S_{3-i}| \geq |(S_i - S_{3-i}) \cap X| = |T_i - T_{3-i}| = k$$

for $i = 1, 2$, so $\min\{|S_1 - S_2|, |S_2 - S_1|\} \geq k$, a contradiction. The statement for $(2,k)$-costars follows since a $(2,k)$-costar is a $(2,k)$-star.

Now suppose $\mathcal{H}$ contains a $(k+1)$-double-chain trace induced by X with trace set $T_1, \ldots, T_{k+1}$. By the definition of the double-chain, $|T_{k+1} - T_1| = |T_1 - T_{k+1}| = k$, so if S_i is an ancestor in $\mathcal{H}$ of T_i, then as above $\min\{|S_1 - S_{k+1}|, |S_{k+1} - S_1|\} \geq k$, a contradiction. Hence $\mathcal{H}$ has no $(k+1)$-double-chain as a trace. $\square$

Proof of Theorem 1. Suppose, toward a contradiction, that G has a k-MDR; call it $\mathcal{S} = \{S(v) : v \in V(G)\}$. Note that $S(u) = S(v)$ iff $u = v$, since with high probability no two distinct vertices of G have the same neighborhood (by Fact 5).

By Fact 4, with high probability, G contains an independent set $I = \{v_1, \ldots, v_{|I|}\}$ with $|I| = \sqrt{\log(n)}$. Let $\mathcal{S}_I = \{S(v_i) : v_i \in I\}$.

Suppose $\mathcal{S}_I$ contains a chain of length 3. Without loss of generality $S(v_1) \subset S(v_2) \subset S(v_3)$. By Fact 5 there is a vertex x such that $xv_1, xv_3 \in E(G)$ and $xv_2 \notin E(G)$. The fact that $xv_1 \in E(G)$ tells us $|S(v_1) - S(x)| \geq k$, so $|S(v_2) - S(x)| \geq k$ as well. Similarly, $|S(x) - S(v_2)| \geq |S(x) - S(v_3)| \geq k$. But then $xv_2 \in E(G)$, a contradiction.

Therefore by Dilworth's Theorem, there is an antichain in $\mathcal{S}_I$ of size $\sqrt{\log n}/2$. Without loss of generality, call this antichain

$$\mathcal{S}^* = \{S(v_i) : 1 \leq i \leq \sqrt{\log n}/2\}.$$

Note that, for all $S_1, S_2 \in \mathcal{S}^*$, $\min\{|S_1 - S_2|, |S_2 - S_1|\} < k$.

Take n large enough that $\sqrt{\log n}/2 \geq h_{2k}(2, k+1, r)$ with $r = 20k^4(2k+1)$. By Theorem 3, $\mathcal{S}^*$ contains a $(2,k)$-star, a $(2,k)$-costar, a $(k+1)$-double-chain, or an (r,p,q)-system with $p, q \leq k$, L_1 empty, and the L_i's with $i \geq 2$ all empty or all nonempty. By Lemma 6, only the fourth case is possible. Call the sets of this (r,p,q)-system $S(v_1), \ldots, S(v_r)$. Let $V_t = \{v_{20k^4t} + 1, \ldots, v_{20k^4(t+1)}\}$ for $0 \leq t \leq 2k$. Let K, F_i, G_i, and L_i be as in the definition of (r,p,q)-system, so that, for $1 \leq i \leq r$,

$$S(v_i) = K \cup F_i \cup \bigcup_{j \neq i} G_j \cup \bigcup_{j \leq i} L_j.$$

With high probability, we can find a set of vertices $X = \{x_1, \ldots, x_{4k^2}\}$ (applying Fact 5 for each x_i) such that x_i is adjacent to v_j iff, for some $0 \leq t \leq 2k$,

$$5k^2(i-1) + 1 + 20k^4t \leq j \leq 5k^2i + 20k^4t$$

(from here on we only use the v_j's with $j \leq r$) and X is an independent set in G. Observe the following two properties:

(1) for each i and t, $|N_G(x_i) \cap V_t| = 5k^2$,
(2) for each i, j, and t with $i \neq j$, $|N_G(x_i) \cap N_G(x_j) \cap V_t| = 0$.

We consider the two cases depending on whether all the L_j's with $j \geq 2$ are empty or nonempty.

CASE 1: $L_j = \emptyset$ for all $1 \leq j \leq r$. For the proof of this case, we work only with v_j's in V_0.

Fix some $x_i \in X$. Define $f_{i,j} = |F_j \cap S(x_i)|$ and $g_{i,j} = |G_j - S(x_i)|$, and let $f_i = \sum_j f_{i,j}$ and $g_i = \sum_j g_{i,j}$. Let $e_i = |K - S(x_i)|$ and $c_i = |S(x_i) - \bigcup_j S(v_j)|$. Then, for any $v_j \in V_0$,

$$|S(v_j) - S(x_i)| = p - f_{i,j} + e_i + g_i - g_{i,j} \tag{1}$$

and

$$|S(x_i) - S(v_j)| = f_i - f_{i,j} + c_i + q - g_{i,j}. \tag{2}$$

Consider a v_j not adjacent to x_i. By the definition of k-MDR, either
(i) $|S(v_j) - S(x_i)| < k$ or
(ii) $|S(x_i) - S(v_j)| < k$.
We claim there are not both v_{j_1} which satisfies (i) and v_{j_2} which satisfies (ii). Indeed, since (i) holds for v_{j_1}, $g_i \leq |S(v_{j_1}) - S(x_i)| + g_{i,j} < 2k$, and, similarly, since (ii) holds for v_{j_2}, we have $f_i \leq |S(x_i) - S(v_{j_2})| + f_{i,j} < 2k$. Hence, for a fixed i, fewer than $2k$ $f_{i,j}$'s and fewer than $2k$ $g_{i,j}$'s are nonzero (this includes v_j's adjacent to x_i). Then we can find vertices v_{j_3}, v_{j_4} with $f_{i,j_3} = g_{i,j_3} = f_{i,j_4} = g_{i,j_4} = 0$ such that x_i is adjacent to v_{j_3} but not to v_{j_4} (here we use Property (1) from above). But then

$$|S(v_{j_4}) - S(x_i)| = |S(v_{j_3}) - S(x_i)| \geq k$$

$$|S(x_i) - S(v_{j_4})| = |S(x_i) - S(v_{j_3})| \geq k,$$

contradicting that $xv_{j_4} \notin E(G)$. Therefore, for each x_i, either

Alternative 1: each v_j not adjacent to x_i satisfies (i) and not (ii), or
Alternative 2: each v_j not adjacent to x_i satisfies (ii) and not (i).

Now, by the pigeonhole principle, there are either at least $2k$ x_i's such that Alternative 1 holds or at least $2k$ x_i's such that Alternative 2 holds (we could in fact have $2k^2$, but $2k$ suffices for this case). Assume the former, and without loss of generality let these vertices be $X_0 = \{x_1, \ldots, x_{2k}\}$. Delete from V_0 all v_j's for which there is an $x_i \in X_0$ such that $G_j \not\subseteq S(x_i)$, and let V' be the subset of V_0 that remains. Since (i) holds for all $x_i \in X_0$, at most $2k$ v_j's satisfy $G_j \not\subseteq S(x_i)$ for each x_i, so

$$|V_0 - V'| \leq 2k|X_0| = 4k^2.$$

Therefore $|N_G(x_i) \cap V'| \geq k^2$ for all $x_i \in X_0$.

Fix some $x_i \in X_0$. Since Alternative 1 holds for x_i, we see $x_i v_j \notin E(G)$ if and only if $|S(v_j) - S(x_i)| < k$. But, for all $v_j \in V'$, $|S(v_j) - S(x_i)| = p - f_{i,j} + e_i + g_i$ (since $g_{i,j} = 0$, by definition of V'), and among the terms on the right, only $f_{i,j}$ depends on j. Therefore there is a threshold $t(x_i)$ such that if $v_j \in V'$, then x_i is adjacent to v_j iff $f_{i,j} \leq t(x_i)$. Since each $x_i \in X_0$ is nonadjacent to some $v_j \in V'$, we deduce that $t(x_i) < p \leq k$.

Since $|X_0| > k$, there are two elements of X_0, without loss of generality x_1 and x_2, with $t(x_1) = t(x_2)$. Now, for all j such that $v_j \in V' \cap N_G(x_1)$ (so $v_j \notin N_G(x_2)$), $|F_j \cap S(x_1)| \leq t(x_1) = t(x_2) < |F_j \cap S(x_2)|$, so there is an element of F_j in $S(x_2)$ and not in $S(x_1)$. Therefore,

$$|S(x_2) - S(x_1)| \geq |V' \cap N_G(x_1)| \geq k^2.$$

Similarly, $|S(x_1) - S(x_2)| \geq |V' \cap N_G(x_2)| \geq k^2$. Hence $x_1 x_2 \in E(G)$, a contradiction.

The case when at least $2k$ x_i's satisfy Alternative 2 is similar. Call the set of these vertices X_0. We find a $V' \subseteq V_0$ such that for all $x_i \in X_0$ and all $v_j \in V'$, $S(x_i) \cap F_j = \emptyset$ (that is, $f_{i,j} = 0$) and $|V' \cap N_G(x_i)| \geq k^2$. Then for each x_i, there is a threshold $t(x_i)$ such that x_i is adjacent to $v_j \in V'$ iff $g_{i,j} \leq t(x_i)$. We then find two vertices x_1 and x_2 with the same threshold, and necessarily

$$\min\{|S(x_1) - S(x_2)|, |S(x_2) - S(x_1)|\} \geq k^2,$$

so x_1 and x_2 are adjacent in G, a contradiction. This completes Case 1.

CASE 2: $L_1 = \emptyset$ and $L_j \neq \emptyset$ for all $j \geq 2$. Define

$$M_t = \bigcup_{v_j \in V_t} L_j$$

for $0 \leq t \leq 2k$. Now, if $S(x_i)$ contains an element of M_t for each $t \geq k+1$ and $M_t - S(x_i)$ is nonempty for each $t \leq k-1$, then $|S(x_i) - S(v_j)|$ and $|S(v_j) - S(x_i)|$ are at least k for all $v_j \in V_k$, contradicting that x_i has a nonneighbor in V_k. Therefore, for each x_i, one of these cannot happen.

Suppose first that for at least $2k^2$ x_i's, it is not the case that $S(x_i)$ contains an element of M_t for each $t \geq k+1$, so there is a $t_i \geq k+1$ such that $S(x_i)$ contains no element of M_{t_i}. Then there is a set X_0 of $2k$ x_i's each with the same t_i, say $t_i = m$ for these x_i's. Then, if we define $b_i = |\bigcup_{t<m} M_t - S(x_i)|$ and follow the notation of Equations 1 and 2, we have for all $v_j \in V_m$,

$$|S(v_j) - S(x_i)| = p - f_{i,j} + e_i + g_i - g_{i,j} + b_i + \left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha \leq j}} L_\alpha - S(x_i) \right|.$$

But

$$\left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha \leq j}} L_\alpha - S(x_i) \right| \geq \left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha \leq j}} L_\alpha \right| \geq j - 20k^4 m$$

since L_j is nonempty for $v_j \in V_m$ and $S(x_i) \cap M_m = \emptyset$. Therefore $|S(v_j) - S(x_i)| \geq k$ for all but the k smallest-indexed v_j's in V_m; let V^* be the set of $v_j \in V_m$ with $|S(v_j) - S(x_i)| \geq k$. Observe that, for all $v_j \in V^*$ and $x_i \in X_0$,

$$|S(x_i) - S(v_j)| = f_i - f_{i,j} + c_i + q - g_{i,j} + s_i + \left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha > j}} L_\alpha \cap S(x_i) \right|,$$

where $s_i = |\bigcup_{t>m} M_t \cap S(x_i)|$. Fix $x_i \in X_0$ and let $v_j \in V^*$ be a nonneighbor of x_i, so necessarily $|S(x_i) - S(v_j)| < k$. Since $g_{i,j} \leq q$, it follows that, $f_i \leq |S(x_i) - S(v_j)| + f_{i,j} \leq 2k - 1$. Therefore, for each i, at most $2k - 1$ $f_{i,j}$'s are nonzero. Delete from V^* each v_j for which there is an $x_i \in X_0$ such that $f_{i,j} \neq 0$, and call the remaining v_j's V'. Then

$$|V^* - V'| \leq (2k - 1)|X_0| \leq 4k^2 - 2k,$$

so each $x_i \in X_0$ has at least $5k^2 - |V_m - V^*| - (4k^2 - 2k) \geq k^2$ neighbors in V'. Since $f_{i,j} = 0$ for all $x_i \in X_0$ and $v_j \in V'$, we see, for a fixed i, $|S(x_i) - S(v_j)|$ depends only on $g_{i,j}$. Thus there is a threshold $t(x_i) \leq k$ such that v_j is adjacent to x_i iff $g_{i,j} \leq t(x_i)$. Since $|X_0| = 2k$, there are two elements of X_0, say x_1 and x_2, with the same threshold. As in the proof of Case 1, it can be proved that x_1 and x_2 are adjacent, a contradiction.

Finally, we are left with the case that there are at least $2k^2$ x_i's such that $M_t - S(x_i)$ is empty for some $t \leq k - 1$. This case is similar to the previous case, except that the roles of $|S(x_i) - S(v_j)|$ and $|S(v_j) - S(x_i)|$ are reversed; we skip the details. □

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