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DELTA-SYSTEM METHODS IN CONTEMPORARY GRAPH THEORY

BY

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*To Ray and Della,*

*And To Mom,*

*To Gracie and Duke,*

*And most importantly, To Asher*

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# Chapter 1

## Introduction

### 1.1 Introduction

In this Dissertation we consider a variety of problems from several areas of contemporary graph theory, namely graph minors, domination, and graph representations.

Graph minors are related to two of the most classical topics in graph theory: colorings and embeddings. The Hadwiger Conjecture asserts a very clear relationship between the chromatic number of a graph  $G$  and the minors of  $G$ : every  $k$ -chromatic graph has a  $K_k$ -minor. The Graph Minor Theorem of Robertson and Seymour (or more accurately the work leading up to the GMT) gave a detailed structural description of graphs embeddable on a given surface in terms of their minors.

A dominating set in a graph is a set of vertices such that every vertex outside the set has a neighbor in the set. Domination is the subdiscipline of graph theory that studies the sizes and structures of dominating sets. Many varieties of domination problems exist, each imposing different conditions on the dominating set itself or on how it interacts with the rest of the vertices. These variations are often used to model communication in networks or in facilities location, making domination a very applied subdiscipline of graph

theory.

Graph representations are ways to describe the vertices of a graph as sets of a given type with adjacency defined as a relation on these sets. For example, vertices may be assigned intervals on the real line and two vertices would be adjacent if and only if their intervals intersect. Questions in this field ask when such representations are possible, and if so, what the “most efficient” representations are. The existence of certain types of representations for classes of graphs has been exploited for such purposes as developing fast graph algorithms.

The work in this dissertation is not unified by topic, but rather by methods employed. All chapters make use of very traditional combinatorial reasoning. Also, all three chapters employ probabilistic techniques, which in the last few decades have become very important throughout graph theory and combinatorics. The more novel aspect of the methods, though, is the use of  $\Delta$ -systems (sometimes called sunflowers). Applications of  $\Delta$ -system methods have seen limited use for quite some time, notably by Frankl and Füredi in Turán theory, but have seen almost no use in many areas of graph and hypergraph theory.

The goal of this Dissertation is to demonstrate the effectiveness and versatility of  $\Delta$ -system methods. A  $\Delta$ -system itself is a set system  $A_1, \dots, A_\ell$  such that there is a set  $K$  with  $A_i \cap A_j = K$  whenever  $i \neq j$ . The  $\Delta$ -system methods we employ involve associating sets with the vertices of  $G$ , finding a  $\Delta$ -system among those sets, and then using the  $\Delta$ -system to locate favorable configurations within the graph  $G$ . Often the set we associate with a vertex is its neighborhood, although when studying graph representations it makes more sense to use the set assigned to it by the representation.

Chapter 2 presents joint work with A.V. Kostochka, in which we seek to show that a dense enough graph  $G$  must have unbalanced complete bipartite minors. A *minor* of a graph  $G$  is a graph which can be obtained from  $G$  through a sequence of vertex deletions, edge deletions, and edge con-

tractions. Kostochka [32, 33] and Thomason[46] independently showed that every graph with average degree  $c \cdot k \log k$  has a  $K_k$ -minor for some constant  $c$ . Thomason [47] determined the smallest possible value of  $c$ . Building on this work, Myers and Thomason [42] determined the average degree that forces a graph to contain an  $H$ -minor (that is, a graph contractible to  $H$ ) for almost all graphs  $H$ . Their results relied on the density of  $H$  being bounded from below by a small constant  $\beta$ . An example of an  $H$  whose density is not bounded below by  $\beta$  is  $K_{s,t}$  for  $t$  sufficiently large (with respect to  $s$ ). In light of this, Myers [41] determined the smallest average degree that forces a graph to contain a  $K_{2,t}$ -minor for all  $t > 10^{29}$  (this was recently extended to all  $t$  by Chudnovsky, Reed, and Seymour [18]). With Kostochka, we determined asymptotically (in  $s$ ) the average degree that forces a  $K_{s,t}$ -minor.

**Theorem 1.1.1** *Let  $s$  and  $t$  be positive integers with*

$$t > (240s \log_2 s)^{8s \log_2 s + 1}.$$

*Let  $G$  be a graph such that  $e(G) \geq \frac{t+3s}{2}(n(G) - s + 1)$ . Then  $G$  has a  $K_{s,t}$ -minor. Furthermore, for  $n$  large, there exists a graph  $G$  of order  $n$  and size at least  $\frac{t+3s-5\sqrt{s}}{2}(n - s + 1)$  that has no  $K_{s,t}$ -minor.*

In the case  $s = 3$ , we can determine exactly the smallest average degree that forces a  $K_{s,t}$ -minor.

**Theorem 1.1.2** *Let  $t \geq (2 \cdot 33! \cdot 6^{33})^2$ . Let  $G$  be a graph of order  $n$  with*

$$e(G) > \frac{1}{2}(t + 3)(n - 2) + 1.$$

*Then  $G$  has a  $K_{3,t}$ -minor.*

A construction due to Myers [41] shows that the edge bound of Theorem 1.1.2 is sharp.

The proofs of the last two theorems are very similar. When the number of vertices  $G$  is not too small, probabilistic and combinatorial reasoning suffice.

For  $G$  of small order, though, we use probability and  $\Delta$ -systems to complete the arguments.

A  $k$ -dominating set of a graph  $G$  is a set  $D \subseteq V(G)$  such that every vertex of  $V(G) - D$  has  $k$  neighbors in  $D$ . The  $k$ -domination number of  $G$ , written  $\gamma_k(G)$ , is the smallest size of a  $k$ -dominating set of  $G$ . When  $k = 1$ , we simply write  $\gamma(G)$  instead of  $\gamma_1(G)$ , and refer to this as the *domination number* of  $G$ . In Chapter 3, we develop techniques for proving Nordhaus-Gaddum bounds on the domination number  $\gamma(G) = \gamma_1(G)$  of a graph, where a *Nordhaus-Gaddum bound* for a parameter  $P$  is any bound on  $P(G) + P(\overline{G})$  or  $P(G) \cdot P(\overline{G})$  ( $\overline{G}$  denotes the complement of the graph  $G$ ). This method, which appears in Section 3.2, is joint work with E.W. Chambers, W. Kinnersley, and D.B. West [16]. In Section 3.3, we generalize the method to prove the following Nordhaus-Gaddum bounds for  $\gamma_k$ .

**Theorem 1.1.3** *For any integer  $k \geq 2$  and any graph  $G$  of order  $n \geq n_0(k)$ ,*

$$8k/3 \leq \gamma_k(G) + \gamma_k(\overline{G}) \leq n + 2k - 1$$

$$16k^2/9 \leq \gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2).$$

*Furthermore, these bounds are sharp.*

The techniques do not generalize in a straightforward manner. When the vertices of  $G$  have large degree, our proofs mimic the techniques for  $\gamma$ , making use of probabilistic and combinatorial arguments. The case when the degrees of  $G$  are small requires  $\Delta$ -systems.

In Section 3.4, we demonstrate the effectiveness of the methods of Section 3.2 by using them to derive Nordhaus-Gaddum bounds for three parameters of the same order of magnitude of  $\gamma$ .

Define a *Roman dominating function* (RDF) of a graph  $G$  to be a map  $f: V(G) \rightarrow \{0, 1, 2\}$  such that every vertex mapped to 0 has a neighbor mapped to 2. Let  $\gamma_R(G)$  denote the minimum over all RDFs  $f$  of

$\sum_{v \in V(G)} f(v)$ . It is not hard to see that  $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ . In Section 3.5, we study when the Roman domination number is close to  $\gamma(G)$  or  $2\gamma(G)$ , proving the following. Let  $\mathbb{G}_{n,p}$  denote the probability space of all  $n$ -vertex graphs where each edge appears randomly and independently with probability  $p$ .

**Theorem 1.1.4** *Let  $p: \mathbb{N} \rightarrow \mathbb{R}$  be a function satisfying  $p \leq 1/2$ . Let  $w = pn$ . Then for almost every  $G \in \mathbb{G}_{n,p}$ ,*

- (1) *if  $w \rightarrow 0$ , then  $\gamma_R(G) = (1 + o(1))\gamma(G)$ ,*
- (2) *if  $w \rightarrow \infty$ , then  $\gamma_R(G) = (2 - o(1))\gamma(G)$ , and*
- (3) *if  $w \rightarrow c \in \mathbb{R}$ , then there is a constant  $c' \in (1, 2)$  such that  $\gamma_R(G) = (c' + o(1))\gamma(G)$ .*

Finally, in Chapter 4, we present joint work with J. Balogh on minimum difference representations of a graph. Define a  $k$ -minimum-difference-representation ( $k$ -MDR) of  $G$  to be an assignment of a finite set  $S(v)$  to each vertex  $v \in V(G)$  such that  $u$  and  $v$  are adjacent if and only if

$$\min(|S(u) - S(v)|, |S(v) - S(u)|) \geq k.$$

Let  $\rho_{\min}(G)$  be the minimum  $k$  for which  $G$  has a  $k$ -MDR. Boros, Gurvich, and Meshulam [13], who introduced minimum difference representations, showed that  $\rho_{\min}(G)$  exists for every finite graph  $G$ . They also noted that no graphs were known for which  $\rho_{\min}(G) > 2$ . The goal of Chapter 4 is to prove the following.

**Theorem 1.1.5** *Let  $k$  be a positive integer. Then  $G \in \mathbb{G}_{n,1/2}$  satisfies  $\rho_{\min}(G) > k$  with high probability as  $n \rightarrow \infty$ .*

The work of Chapter 4 uses  $\Delta$ -system methods in a different way than Chapters 2 and 3. Instead of using  $\Delta$ -systems to find a favorable set of vertices in our graph  $G$ , we use the sets of the  $\Delta$ -system themselves to arrive at a contradiction. Strictly speaking, we do not use  $\Delta$ -systems, but rather a variant of

them, which we call  $(r, p, q)$ -systems. Define a *trace* of a hypergraph  $\mathcal{H}$  to be the hypergraph with vertex set  $X$  and edge set  $\{X \cap E : E \in E(\mathcal{H})\}$ . Define an  $(\ell, k)$ -*star* to be a  $k$ -uniform set system  $A_1, \dots, A_\ell$  such that  $A_i \cap A_j = \emptyset$  if  $i \neq j$ . Define an  $(\ell, k)$ -*costar* to be a set system  $A_1, \dots, A_\ell$  such that, if  $A = \bigcup_{j=1}^{\ell} A_j$ , then  $A - A_1, \dots, A - A_\ell$  is an  $(\ell, k)$ -star. Define an  $m$ -*double-chain* to be a hypergraph with vertices  $v_1, \dots, v_{2m-1}$  and edge set  $\{\{v_{i+1}, \dots, v_{i+m-1}\} : 1 \leq i \leq m\}$ . Our main tool is the following.

**Theorem 1.1.6** *Let  $\ell, k, m,$  and  $r$  be positive integers. Then every antichain  $\mathcal{H}$  with  $|\mathcal{H}| \geq h_{2k}(\ell, m, r)$  contains an  $(\ell, k)$ -star, an  $(\ell, k)$ -costar, or an  $m$ -double-chain as a trace or contains an  $(r, p, q)$ -system with  $p, q \leq k$  as a subsystem.*

Let  $G$  be a random graph in  $\mathbb{G}_{n,1/2}$ . When we apply Theorem 1.1.6 with  $\mathcal{H} = \{S(v) : v \in I\}$  ( $S(v)$  being the set assigned to  $v$  by the  $k$ -MDR) where  $I$  is a maximum independent set of  $G$ , we can immediately rule out the possibility that  $\mathcal{H}$  contains one of the three specified traces. The remainder of the effort goes to showing that  $\mathcal{H}$  also cannot contain an  $(r, p, q)$ -system with  $p, q \leq k$ . This contradiction shows almost every  $G \in \mathbb{G}_{n,1/2}$  has no  $k$ -MDR, so  $\rho_{\min}(G) > k$ .

It is our intention that these proofs will serve as models of how to use  $\Delta$ -systems, as well as inspire more sophisticated applications of the methods.

## 1.2 Notation

Before we begin, let us fix notation. We follow [48], but review the most commonly used terms and symbols.

We assume the axioms of ZFC set theory and the Peano axioms for the set of natural numbers  $\mathbb{N}$ .

A *graph* is an ordered pair  $(V(G), E(G))$ , where  $V(G)$  is the *vertex set* of  $G$  and  $E(G) \subseteq \binom{V(G)}{2}$  is the *edge set* of  $G$ . We use  $n(G)$  for the *order* of

$G$ , defined as  $|V(G)|$ , and  $e(G)$  for the *size* of  $G$ , defined to be  $|E(G)|$ . Two graphs  $G$  and  $H$  are *isomorphic* if there is a bijective function  $f: V(G) \rightarrow V(H)$  such that  $xy \in E(G)$  if and only if  $f(x)f(y) \in E(H)$ . We write  $G \cong H$  to indicate that  $G$  and  $H$  are isomorphic.

A *path* in a graph  $G$  is a sequence of distinct vertices  $v_1, \dots, v_k$  in  $V(G)$  such that  $v_i v_{i+1} \in E(G)$  for all  $i \in \{1, \dots, k-1\}$ . The *length* of the path  $v_1, \dots, v_k$  is  $k-1$ .

A *cycle* in  $G$  is a sequence of distinct vertices  $v_1, \dots, v_k$  such that  $v_i v_{i+1 \pmod k} \in E(G)$  for  $i \in \{1, \dots, k\}$ . The *length* of the cycle  $\{v_1, \dots, v_k\}$  is  $k$ .

The *girth* of a graph  $G$ , written  $\text{girth}(G)$ , is the length of the shortest cycle in  $G$ .

All graphs considered have finite vertex sets.

For a vertex  $v \in V(G)$ , the *degree* of  $v$  in  $G$  is written  $d_G(v)$ . The minimum and maximum degrees of  $G$  are denoted  $\delta(G)$  and  $\Delta(G)$ . We say a graph is *regular* if  $\delta(G) = \Delta(G)$  and  *$k$ -regular* if  $\delta(G) = \Delta(G) = k$ .

A *subgraph* of  $G$  is a graph  $H$  such that  $V(H) \subseteq V(G)$  and  $E(H) \subseteq \binom{V(H)}{2} \cap E(G)$ . An  *$H$ -subgraph* of  $G$  is a subgraph of  $G$  isomorphic to  $H$ . An *induced subgraph* of  $G$  is a subgraph  $H$  for which  $E(H) = \binom{V(H)}{2} \cap E(G)$ . We use  $G[A]$  to denote the *subgraph of  $G$  induced by  $A$* , which is the graph whose vertex set is  $A$  and edge set is  $\binom{A}{2} \cap E(G)$ .

Let  $X$  and  $Y$  be disjoint subsets of  $V(G)$ . Then  $E_G(X, Y)$  denotes the set of edges of  $G$  with one endpoint in  $X$  and the other in  $Y$ . An  *$X, Y$ -path* is a path  $v_1, \dots, v_k$  in  $G$  with  $v_1 \in X$  and  $v_k \in Y$ . We say a graph  $G$  is *connected* if for all pairs of disjoint subsets  $X$  and  $Y$  of  $V(G)$ ,  $G$  contains an  $X, Y$ -path. We define the *distance* between  $X$  and  $Y$  in  $G$ , written  $\text{dist}_G(X, Y)$ , to be the length of the shortest  $X, Y$ -path in  $G$ . A set  $S \subseteq V(G)$  is *connected* if  $G[S]$  is a connected graph.

For a vertex  $x \in V(G)$ ,  $N_G(x)$  is the (*open*) *neighborhood* of  $x$ , which is the set  $\{u \in V(G) : uv \in E(G)\}$ . We use  $N_G[x]$  to denote the *closed*

*neighborhood*, defined as  $N_G(x) \cup \{x\}$ . For a set  $X \subseteq V(G)$ , we write

$$N_G^k(X) = \{v \in V(G) - X : |N_G(v) \cap X| \geq k\}$$

and  $N_G^k[X] = N_G^k(X) \cup X$  (if  $k = 1$  we omit the superscript). We say that  $X$  *k-dominates*  $Y$  if  $Y \subseteq N_G^k[X]$  (for  $k = 1$  we say simply “dominates”). A *dominating vertex* of  $G$  is a vertex  $v \in V(G)$  such that  $N[v] = V(G)$ . We use  $\mathcal{N}_G$  to denote the *neighborhood hypergraph* of  $G$ , whose vertex set is  $V(G)$  and whose set of hyperedges is  $\{N_G(v) : v \in V(G)\}$ , where hyperedges appear with multiplicity.

A set  $S \subseteq V(G)$  is said to be *independent* if  $e(G[S]) = 0$ . A *bipartite graph* is a graph  $G$  such that  $V(G)$  can be written as the disjoint union of independent sets  $X$  and  $Y$ . We call  $X$  and  $Y$  the *partite sets* of  $G$ .

For two graphs  $G$  and  $H$ , define the graph  $G \vee H$ , called “ $G$  join  $H$ ”, as follows: let  $G'$  and  $H'$  be isomorphic to  $G$  and  $H$ , respectively, such that  $V(G')$  and  $V(H')$  are disjoint. Let  $V(G \vee H) = V(G') \cup V(H')$  and  $E(G \vee H) = E(G') \cup E(H') \cup \{xy : x \in V(G'), y \in V(H')\}$ .

For all graph parameters, we reserve the right to omit the subscript “ $G$ ” if the context is clear.

For two sets  $A$  and  $B$ ,  $A \subseteq B$  means that  $A$  is a subset of  $B$  and  $A \subset B$  means  $A$  is a proper subset of  $B$ . A *set system* is a family of finite sets. A set system  $\mathcal{H}$  is an *antichain* if there are no distinct  $A, B \in \mathcal{H}$  with  $A \subset B$ .

We use  $\mathbb{G}_{n,p}$  to denote the probability space of all  $2^{\binom{n}{2}}$  graphs on a fixed set of  $n$  labelled vertices where each potential edge appears independently with probability  $p$ . Therefore  $\mathbb{G}_{n,1/2}$  is the space of  $n$ -vertex labelled graphs (on a fixed vertex set) with uniform distribution. We use  $\mathbb{P}$  and  $\mathbb{E}$  to denote the probability and expectation operators. We will use  $\mathbb{E}_A(X)$  to denote the expectation of the random variable  $X$  over the set  $A$ . We say that event  $E$  happens *with high probability* if the probability of  $E$  in  $\mathbb{G}_{n,p}$  approaches 1 as  $n \rightarrow \infty$ . We also use the language “*almost every*  $G \in \mathbb{G}_{n,p}$  satisfies  $E$ ” or “ $E$  happens *almost always*” in place of “ $E$  happens with high probability”.

Any additional notation will be defined as it is needed.

# Chapter 2

## Graph Minors

### 2.1 Introduction

A *minor* of a graph  $G$  is a graph  $H$  that can be obtained from  $G$  by a sequence of vertex and edge deletions and edge contractions. We say a subgraph  $F$  of  $G$  is an  *$H$ -minor* of  $G$  if  $H$  can be obtained from  $F$  by a sequence of edge contractions, and we write  $H \prec G$  if  $G$  has an  $H$ -minor.

A well-known open problem concerning graph minors is the Hadwiger Conjecture.

**Conjecture 2.1.1** [*Hadwiger, 1943*] *Every  $k$ -chromatic graph has a  $K_k$ -minor.*

Conjecture 2.1.1 is known to be true for  $k \leq 6$  but remains open for all larger values of  $k$ . In the positive direction, Bollobás, Catlin, and Erdős [12] showed Conjecture 2.1.1 is true for almost all graphs (that is, almost every  $k$ -chromatic graph has a  $K_k$ -minor).

One way to generalize this problem is to search for other sufficient conditions that force a graph to contain a  $K_k$ -minor. Mader [38] proved that for each positive integer  $k$ , there exists a function  $D(k)$  such that every graph with average degree at least  $D(k)$  has a  $K_k$ -minor by demonstrating that

$D(k) \leq 2^{\binom{k}{2}+1}$ . Later, Kostochka [32, 33] and Thomason [46] determined that  $D(k) \in \Theta(k\sqrt{\log k})$ , with the lower bound coming from almost every  $G \in \mathbb{G}_{n,1/2}$ . More recently, Thomason [47] determined that

$$D(k) = (\alpha + o(1))k\sqrt{\log k},$$

where  $\alpha = 0.6381726\dots$  is given explicitly.

Myers and Thomason extended the function  $D$  above to general graphs  $H$ , that is, they defined

$$D(H) = \inf\{d \mid 2e(G)/n(G) \geq d \text{ implies } H \prec G\},$$

which is essentially the smallest average degree that forces a graph to contain an  $H$ -minor (where  $D(K_k)$  equals  $D(k)$  from above). They determined  $D(H)$  for almost every  $H$  [42, 39], showing, in particular, that for almost all  $H$ , the extremal graphs not containing  $H$  are pseudo-random (built deterministically from randomly generated subcomponents). However, their methods relied on the density of  $H$  (with *density* defined, naturally, to be  $e(H)\binom{n(H)}{2}^{-1}$ ) being bounded away from 0.

An example of an  $H$  whose density is not bounded away from 0 is  $K_{s,t}$  where  $s$  is fixed and  $t$  is sufficiently large with respect to  $s$ . For this reason, Myers [40, 41] studied  $D(K_{s,t})$  when  $s$  is fixed and  $t$  is large. Let  $M(r, s, t)$  be the graph obtained by taking  $r$  copies of  $K_{s+t-1}$  and sharing a common set of  $s-1$  vertices (this graph can also be described as  $K_{s-1} \vee rK_t$ ). Myers [41] noted that  $M(r, s, t)$  has no  $K_{s,t}$ -minor and has  $\frac{1}{2}(t+2s-3)(n-s+1) + \binom{s-1}{2}$  edges, where  $n = r(t+1) + s - 1$ . He proved the following.

**Theorem 2.1.2** ([41]) *Let  $t > 10^{29}$  be a positive integer. Then every graph  $G$  with more than  $\frac{t+1}{2}(n(G) - 1)$  edges has a  $K_{2,t}$ -minor.*

Myers' construction ( $M(r, s, t,)$  of the last paragraph) shows that this bound is sharp when  $n(G) \equiv 1 \pmod{t}$ .

Very recently, Chudnovsky, Reed, and Seymour [18] proved that Theorem 2.1.2 in fact holds for all  $t$ .

Myers conjectured that a similar, more general statement was true for  $K_{s,t}$ -minors.

**Conjecture 2.1.3** *Let  $s$  be a positive integer. Then there exists a constant  $C(s)$  such that, for all positive integers  $t$ , if  $G$  has average degree at least  $C(s) \cdot t$ , then  $K_{s,t} \prec G$ .*

Further, if we define  $K_{s,t}^*$  to be  $K_s \vee \overline{K_t}$ , Myers noted that the average degree that forces  $G$  to contain a  $K_{s,t}$ -minor also likely forces a  $K_{s,t}^*$ -minor, that is,  $D(K_{s,t}) = D(K_{s,t}^*)$  (trivially we have  $D(K_{s,t}) \leq D(K_{s,t}^*)$ , since  $K_{s,t} \subseteq K_{s,t}^*$ ).

For our purposes, it will be more convenient to work with the quantity  $D^*(K_{s,t})$ , defined as

$$D^*(K_{s,t}) = \inf\{d \mid 2e(G)/(n(G) - s + 1) \geq d \text{ implies } H \prec G\},$$

than with  $D(K_{s,t})$  itself.

In Section 2.2, we prove the following theorem, establishing Conjecture 2.1.3 and confirming Myers' insight about  $D(K_{s,t}^*)$ .

**Theorem 2.1.4** *Let  $s$  and  $t$  be positive integers with*

$$t > (240s \log_2 s)^{8s \log_2 s + 1}.$$

*Let  $G$  be a graph such that  $e(G) \geq \frac{t+3s}{2}(n(G) - s + 1)$ . Then  $G$  has a  $K_{s,t}^*$ -minor. Furthermore, for  $n$  large, there exists a graph  $G$  of order  $n$  and size at least  $\frac{t+3s-5\sqrt{s}}{2}(n - s + 1)$  that has no  $K_{s,t}$ -minor.*

Theorem 2.1.4 will be established in two steps: the lower bound will come from Theorem 2.2.2 in Subsection 2.2.1 and the upper bound will be from Theorem 2.2.15 in Subsection 2.2.4.

From Theorem 2.1.4 we have the following string of inequalities:

$$t + 3s - 5\sqrt{s} \leq D^*(K_{s,t}) \leq D^*(K_{s,t}^*) \leq t + 3s.$$

Hence, Myers' insight that  $D(K_{s,t})$  is the same as  $D(K_{s,t}^*)$  is true asymptotically in  $s$ . However,  $D(K_{s,t}) \neq D(K_{s,t}^*)$  in general, as an example in Subsection 2.3 shows (at least for  $s = 3$ ).

For  $s > 100$ , the second half of Theorem 2.1.4 tells us that Myers' construction giving  $D^*(K_{s,t}) \geq t + 2s - 3$  is not optimal. In fact, we provide a second construction with fewer edges which shows that Myers' construction  $M(r, s, t)$  is not optimal for  $s \geq 6$ .

We note the following result of Kühn and Osthus, proved independently of and concurrently to our Theorem 2.1.4.

**Theorem 2.1.5** ([36]) *For every  $\epsilon > 0$  and every positive integer  $s$ , there exists a number  $t_0 = t_0(s, \epsilon)$  such that for all  $t \geq t_0$ , every graph of average degree at least  $(1 + \epsilon)t$  contains  $K_{s,t}^*$  as a minor.*

While Theorem 2.1.5 does not provide the dependence of  $D(K_{s,t})$  on  $s$ , it does apply for a much wider range of  $s$ , namely  $s \leq C \cdot t / \log t$ .

In the case  $s = 3$ , Kostochka and Prince [35] determine  $D(K_{s,t})$  exactly.

**Theorem 2.1.6** *Let  $t \geq (2 \cdot 33! \cdot 6^{33})^2$ . Let  $G$  be a graph of order  $n$  with*

$$e(G) > \frac{1}{2}(t + 3)(n - 2) + 1.$$

*Then  $G$  has a  $K_{3,t}$ -minor.*

Myers' construction demonstrates the sharpness of Theorem 2.1.6.

## 2.2 $K_{s,t}$ -minors

In this section, we shall prove Conjecture 2.1.3. We begin in Subsection 2.2.1 by exhibiting constructions showing  $D^*(K_{s,t}) \geq t + 3s - 5\sqrt{s}$ . We then provide an upper bound of  $t + 3s$  for  $D^*(K_{s,t}^*)$  when  $t$  is sufficiently large with respect to  $s$ .

### 2.2.1 The Lower Bound

For this subsection, it will be convenient to use the following equivalent definition of a  $K_{s,t}$ -minor of  $G$ . We say  $G$  has a  $K_{s,t}$ -minor if there are a set  $V_0 \in V(G)$  and a function  $f: (V(G) - V_0) \rightarrow V(K_{s,t})$  such that  $f^{-1}(v)$  is a connected subset of  $G$  for all  $v \in V(K_{s,t})$  and such that, for all  $v_1v_2 \in E(K_{s,t})$ , there is an  $x_i \in f^{-1}(v_i)$  ( $i = 1, 2$ ) such that  $x_1x_2 \in E(G)$ . In this language,  $G[V(G) - V_0]$  is contractible to  $K_{s,t}$ , so it is our  $K_{s,t}$ -minor.

We will need the following old result of Sauer [45]:

**Theorem 2.2.1** *Let  $g \geq 5$  and  $m \geq 4$ . Then, for every even*

$$n \geq 2(m-1)^{g-2},$$

*there exists an  $n$ -vertex  $m$ -regular graph of girth at least  $g$ .*

**Theorem 2.2.2** *Let  $s \geq 3$  and  $t > 2s^{2s-1}$  be integers. Then there exists a graph  $G(s, t)$  with*

$$e(G(s, t)) \geq \frac{1}{2}(t + 3s - 5\sqrt{s})(n(G(s, t)) - s + 1)$$

*and no  $K_{s,t}$ -minor. Therefore  $D^*(K_{s,t}) \geq t + 3s - 5\sqrt{s}$ .*

*Proof.* We may assume  $s \geq 19$ , since otherwise  $t + 2s - 3 \geq t + 3s - 5\sqrt{s}$ , and so Myers' construction is our  $G(s, t)$ .

Let  $q$  be the positive integer for which  $\lceil \sqrt{3s} \rceil \leq q \leq \lceil \sqrt{3s} + 1 \rceil$  and  $t - q$  is even. Note that

$$2.5\sqrt{s} \geq \lceil \sqrt{3s} \rceil + 1 \geq q$$

and  $s \geq q \geq 6$  since  $s \geq 19$ .

By Theorem 2.2.1, since  $t + 2s - q > 2(q - 3)^{2s-1}$ , there is a  $(q - 2)$ -regular graph of girth at least  $2s + 1$  with  $n = t + 2s - q$  vertices; let this graph be  $H(s, t)$ .

First, we claim that

$$e(\overline{H(s,t)}) \geq \frac{1}{2}(t+3s-2q)(n-s+1).$$

This is because

$$e(\overline{H(s,t)}) = \frac{1}{2}n(n-1-(q-2)) = \frac{1}{2}(t+2s-q)(t+2s-2q+1),$$

and since

$$(t+2s-q)(t+2s-2q+1) \geq (t+3s-2q)(t+s-q+1) = (t+3s-2q)(n-s+1)$$

(this inequality is equivalent to  $(s-1)(s-q) \geq 0$ ), the claim follows.

**Claim 1**  $\overline{H(s,t)}$  has no  $K_{s,t}$ -minor.

*Proof of Claim.* Suppose a  $V_0 \subseteq V(H(s,t))$  and a function  $f$  as in the definition of a minor (from the beginning of the subsection) existed. Let  $X \subseteq V(K_{s,t})$  be the set of vertices  $x$  such that  $|f^{-1}(x)| \geq 2$ . Let  $W = V_0 \cup f^{-1}(X)$ . Then

$$t+2s-q = n = n(K_{s,t}) + \sum_{v \in X} (|f^{-1}(v)| - 1) + |V_0| =$$

$$t+s+|f^{-1}(X)| - |X| + |V_0| \geq t+s+(|f^{-1}(X)| + |V_0|)/2,$$

so  $|W| \leq 2(s-q)$ .

Let  $S$  denote the partite set of  $s$  vertices in  $K_{s,t}$  and let  $P = f^{-1}(S-X) = f^{-1}(S) - W$ . Then  $|P| \geq q$ . Since, in  $\overline{H(s,t)}$ , every vertex in  $P$  is adjacent to every vertex outside of  $P \cup W$ , we see that  $H(s,t)[P \cup W]$  contains all edges in  $H(s,t)$  incident to a vertex in  $P$ . But we know that

$$\text{girth}(H(s,t)) \geq 2s+1 \geq |P|,$$

so  $e_{H(s,t)}(P) \leq |P| - 1$ . Therefore, there are at least  $(q-2)|P| - (|P| - 1)$  edges of  $H(s,t)$  incident to  $P$ ; let  $H_0$  be the subgraph induced by these edges.

If  $H_0$  has a cycle, then at least half the vertices of this cycle are in  $P$ , so this cycle has length at most  $2|P| \leq 2s$ , contradicting the definition of  $H(s, t)$ . Since  $H_0$  is acyclic,

$$|P \cup W| \geq (q-2)|P| - (|P| - 1) + 1 \geq (q-3)|P| + 2.$$

However, we also know that  $|P \cup W| \leq |P| + 2(s - q)$ , and so

$$2(s - q) \geq (q-3)|P| + 2 \geq (q-3)q + 2,$$

that is,  $2s \geq q^2 - q + 2$ . But this does not hold if  $s \geq 19$  and  $q \geq \sqrt{3s}$ .  $\square$

Letting  $G(s, t) = \overline{H(s, t)}$  proves the theorem.  $\blacksquare$

The graph  $G(s, t)$  from the proof of Theorem 2.2.2 is already enough to show  $D^*(K_{s,t}) \geq t + 3s - 5\sqrt{s}$ . However,  $n(G(s, t))$  is quite small compared to  $t$ . We can construct graphs with more vertices and the same asymptotic density by gluing copies of  $G(s, t)$  together at a set  $I$  of  $s-1$  vertices inducing a clique. This process cannot create a  $K_{s,t}$ -minor: if there were such a minor, then there would be a  $v \in S$  such that  $f^{-1}(v)$  lies completely outside of  $I$ . Then, since there are no  $s$  vertex-disjoint paths from  $f^{-1}(v)$ , all of  $f^{-1}(T)$  lies in the same copy of  $G(s, t)$  as  $f^{-1}(v)$ . Similarly, all of  $f^{-1}(S)$  lies in the same copy. Now this copy contains a  $K_{s,t}$ -minor, which is absurd.

When  $6 \leq s < 19$ , there are graphs with fewer edges than  $G(s, t)$  that nonetheless show that Myers' construction is not optimal.

**Theorem 2.2.3** *Let  $s$  and  $t$  be integers satisfying  $t > (2s)^{2s-1}$  and  $s \geq 6$ . Then there exists a graph  $G$  of order  $n = t + s + 3$  with*

$$e(G) \geq \frac{1}{2}n \left( t + s - 2 + \frac{1}{2 \cdot 3^{s+7}} \right)$$

*that contains no  $K_{s,t}$ -minor.*

*Proof.* Let  $H$  be a 4-regular graph with girth at least  $s + 7$ , which exists by Theorem 2.2.1. Let  $A$  be a set of at least  $n/2 \cdot 3^{s+7}$  edges at distance at least  $s + 7$  from each other (such an  $A$  can be found greedily). Let  $H' = H - A$ .

Let  $r$  satisfy  $s - 3 \leq r \leq s$ . If  $U$  is any set of  $r$  vertices of  $H'$  with  $H'[U]$  having  $k$  components and  $m$  vertices of degree 3, then we have  $e_{H'}(U, \overline{U}) = 2r - m + 2k$ . Suppose  $|N_{H'}(U) - U| \leq 6$ . Since each vertex with  $d$  neighbors in  $U$  links together  $d$  components of  $G[U]$ , if

$$|E_{H'}(U, V(H') - U)| = \sum_{v \in N_{H'}(U) - U} (|N_{H'}(v) \cap U| - 1) \geq k - m + 1,$$

then  $U \cup N_{H'}(U)$  contains a cycle of length at most  $s + 7$  or two vertices of degree 3 at distance at most  $s + 6$ , either way a contradiction. Therefore, since  $|N_{H'}(U) - U| \leq 6$ , we have  $2r - m + 2k \leq k - m + 6$ , so  $2r + k \leq 6$ . Thus  $3 > r \geq s - 3 \geq 3$ . This contradiction implies that any set  $U$  with  $s - 3 \leq |U| \leq s$  satisfies  $|N_{H'}(U)| \geq 7$ .

Let  $G = \overline{H'}$ . Suppose  $G$  has a  $K_{s,t}$ -minor. Let  $S \subseteq V(G)$  be the set of vertices in the preimage of the smaller partite set of this minor, and let  $S' \subseteq S$  be the vertices which are not deleted or contracted into a neighbor to get the minor. Applying the argument of the last paragraph to  $\overline{G}$  (with  $U = S'$ ), there must be at least 7 vertices with a nonneighbor in  $S'$ , and at least one of these vertices  $x$  is the entire preimage of a vertex of the larger partite set in the minor. This contradicts that every vertex of  $S'$  is adjacent to  $x$ . Therefore  $G$  has no  $K_{s,t}$ -minor.  $\blacksquare$

When  $n > 16 \cdot 3^{s+7}$ , the graph of Theorem 2.2.3 contains more edges than Myers construction.

## 2.2.2 Small Graphs

In this section we prove Theorem 2.1.4 in the case when  $n(G) \leq 10t/9$ .

We start with a lemma we will apply to the complement of  $G$ .

**Lemma 2.2.4** *Let  $m, s$ , and  $n$  be positive integers such that  $s \geq 3$  and*

$$n > 10s(30m)^m.$$

*Let  $H$  be a graph of order  $n$  with  $e(H) \leq mn/2$  such that  $d_H(v) \leq 3n/5$  for all  $v \in V(H)$ . Then there is an  $L \subset V(H)$  with  $|L| \leq m - 1$  and  $s$  disjoint pairs  $(x_i, y_i)$  of vertices in  $H - L$  such that  $\text{dist}_{H-L}(x_i, y_i) \geq 3$  for all  $i = 1, \dots, s$ .*

*Proof.* For each pair of distinct vertices  $x, y$  in  $H$ , define  $A(x, y)$  to be the set of common neighbors of  $x$  and  $y$ . Let  $a(x, y) = |A(x, y)|$ . If we let  $a(H)$  be  $\sum_{x, y \in V(H)} a(x, y)$ , we have that

$$a(H) = \sum_{x \in V(H)} \binom{d_H(x)}{2} \leq \binom{3n/5}{2} n \leq \binom{3n/5}{2} \frac{mn}{3n/5} < \frac{3}{10} mn^2.$$

Let  $V_{\text{lg}} = \{v \in V(H) \mid d_H(v) \geq n/10m\}$  be the vertices with “large” degree in  $H$ , and define  $V_{\text{sm}} = V(H) - V_{\text{lg}}$  (these have “small” degree). Note that  $|V_{\text{lg}}| \leq 2e(H)/(n/10m) \leq 10m^2$ .

For every pair of vertices  $x, y \in H$ , let  $A_{\text{lg}}(x, y) = A(x, y) \cap V_{\text{lg}}$  and  $a_{\text{lg}}(x, y) = |A_{\text{lg}}(x, y)|$ . Define  $A_{\text{sm}}(x, y)$  and  $a_{\text{sm}}(x, y)$  analogously. As above, define

$$a_{\text{lg}}(H) = \sum_{x, y \in H} a_{\text{lg}}(x, y) \quad \text{and} \quad a_{\text{sm}}(H) = \sum_{x, y \in H} a_{\text{sm}}(x, y).$$

Similar to the calculation in the last paragraph,

$$a_{\text{sm}}(H) \leq \sum_{x \in V_{\text{sm}}} \binom{d_H(x)}{2} \leq \binom{n/10m}{2} \frac{mn}{n/10m} < n^2/20.$$

Let  $W = \{(x, y) \in \binom{V(H)}{2} \mid xy \notin E(H), a_{\text{sm}}(x, y) = 0, \text{ and } a_{\text{lg}}(x, y) \leq m - 1\}$ . Now clearly the number of pairs  $(x, y)$  with  $a_{\text{sm}}(x, y) = 0$  is at most  $a_{\text{sm}}(H)$ . Also, the number of pairs  $(x, y)$  with  $a_{\text{lg}}(x, y) \geq m$  is at most  $a(H)/m$ . We therefore have

$$|W| \geq \binom{n}{2} - e(H) - a_{\text{sm}}(H) - \frac{a(H)}{m}$$

$$\geq \binom{n}{2} - \frac{mn}{2} - \frac{n^2}{20} - \frac{3n^2}{10} = \frac{n}{2} \left( \frac{3n}{10} - m - 1 \right) > \frac{n(n-1)}{9}.$$

Consider the graph  $F$  with vertex set  $V(H)$  and edge set  $W$ . Since  $e(F) > n(n-1)/9$ ,  $F$  has a matching  $M$  of size at least  $n/9$ .

Since the number of distinct subsets of  $V_{\text{lg}}$  of size at most  $m-1$  is

$$\sum_{k=0}^{m-1} \binom{10m^2}{k} < \binom{10m^2}{m} < (10em)^m,$$

by the pigeonhole principle, there is an  $L \subseteq V_{\text{lg}}$  with  $|L| \leq m-1$  such that the set  $M_L = \{xy \in M \mid A_{\text{lg}}(x, y) = L\}$  satisfies

$$|M_L| \geq \frac{n/9}{(10em)^m} > s.$$

But then  $L$  and the pairs of  $M_L$  are what we want. ■

**Lemma 2.2.5** *Suppose that  $t > (240s \log_2 s)^{1+8s \log_2 s}$ . If  $G$  is a graph of order  $n \leq t + 8s \log_2 s$  such that*

$$e(G) > \frac{t+3s}{2}(n-s+1),$$

*then  $G$  has a  $K_{s,t}^*$ -minor.*

*Proof.* Let  $G'$  be a minor-minimal subgraph of  $G$  with respect to the edge condition. Let  $n' = n(G') = t + 2s + m$ . The edge condition forces  $m$  to be positive, so  $1 \leq m \leq 8s \log_2 s$ .

Now

$$e(\overline{G'}) < \binom{n'}{2} - \frac{t+3s}{2}(n'-s+1) \tag{2.1}$$

$$= \frac{1}{2}(n'^2 - n' - (n'+s-m)(n'-s+1)) \tag{2.2}$$

$$= \frac{1}{2}((m-2)n' + (s-1)(s-m)) \tag{2.3}$$

$$< \frac{mn'}{2}. \tag{2.4}$$

Also, by minor-minimality of  $G'$ , the minimum degree of  $G'$  is at least  $(t + 3s)/2$ , so

$$\Delta(\overline{G'}) \leq n' - 1 - (t + 3s)/2 = (t + s)/2 + m - 1 < 3n'/5.$$

Since  $n' > t > (240s \log_2 s)^{1+8s \log_2 s} \geq 10s(30m)^m$ , we can apply Lemma 2.2.4 to  $\overline{G'}$ , finding  $L \subset V(G)$  with  $|L| \leq m - 1$  and  $s$  disjoint pairs of vertices  $(x_i, y_i)$  for  $1 \leq i \leq s$  such that  $\text{dist}_{G-L}(x_i, y_i) > 2$  for all  $i$ . Then contracting the edges  $x_i y_i$  in  $G' - L$ , we find a  $K_{s, n-|L|-2s}^*$ -minor in  $G'$  (and hence in  $G$ ). Since  $n - |L| - 2s \geq t$ , we are done.  $\blacksquare$

**Lemma 2.2.6** *Let  $s, t$ , and  $d$  be positive integers satisfying  $s \geq 3$ ,  $t \geq 4000s^2$ ,  $d \geq 8s \log_2 s$  and  $d \leq t/9$ . Let  $G$  be a graph of order  $t + d$  with minimum degree at least  $t/2$  and at least  $s$  vertices with degree at least  $t$ . Then  $G$  contains a  $K_{s,t}^*$ -minor.*

*Proof.* Let  $v_1, \dots, v_s$  have degree at least  $t$ . Let  $k = \lceil \log_{3/2} d \rceil + 1$ . We will find  $s$  disjoint dominating sets  $S_i$  with  $|S_i| \leq k$  for each  $1 \leq i \leq s$ .

Initialize  $S_i^0 = \{v_i\}$  for  $1 \leq i \leq s$ . For each  $1 \leq i \leq s$ , run the following procedure. Define  $U_i^j$  be the set of vertices not dominated by  $S_i^j$ . If  $U_i^j$  is empty, set  $S_i = S_i^j$  and stop. Otherwise, define  $W_i^j = V(G) - U_i^j - \bigcup_{q < i} S_q$ . Since

$$|N_G(u) \cap W_i^j| \delta(G) - |U_i^j| - (i - 1)k \geq 2t/5 - (i - 1)k$$

for all  $u \in U_i^j$ , there must be a vertex  $v_i^j \in W_i^j$  such that

$$|N_G(v_i^j) \cap U_i^j| \geq \frac{(2t/5 - (i - 1)k)|U_i^j|}{t + d - |U_i^j|} \geq \frac{(2t/5 - (s - 1)k)|U_i^j|}{11t/10} \geq \frac{|U_i^j|}{3}.$$

Let  $S_i^{j+1} = S_i^j \cup \{v_i^j\}$  and iterate the process.

At each step,  $|U_i^j| \leq (2/3)^j d$ , so  $U_i^k$  is empty for  $1 \leq i \leq s$ . This means  $S_i$  is a dominating set of  $G$  with size at most  $k + 1$ . Also,  $S_i$  is connected because each  $v_i^j$  was chosen among the neighbors of  $S_i^j$ . Since  $s(k + 1) \leq d$ ,

contracting each  $S_i$  to a single vertex gives a  $K_{s,t}^*$ -minor of  $G$ . ■

**Lemma 2.2.7** *Let  $s$  and  $t \geq (240s \log_2 s)^{8s \log_2 s + 1}$  be positive integers. Let  $G$  be a graph of order  $n$ , where  $t + 2s + 1 \leq n \leq 10t/9$ . Suppose*

$$e(G) > \frac{1}{2}(t + 2s - 3)(n - s + 1).$$

*Then  $G$  contains a  $K_{s,t}^*$ -minor.*

*Proof.* We may assume that  $G$  is minor-minimal with respect to the edge condition. This implies, in particular, that  $G$  has minimum degree at least  $(t + 3s)/2$ .

If  $t + 2s + 1 \leq n \leq t + \lceil 8s \log_2 s \rceil$ , apply Lemma 2.2.5. If  $n > t + \lceil 8s \log_2 s \rceil$ , notice that the edge bound on  $G$  immediately implies that  $G$  has  $s$  vertices of degree at least  $t$ . Therefore we can apply Lemma 2.2.6 to find a  $K_{s,t}^*$ -minor in  $G$ . This completes the proof. ■

**Theorem 2.2.8** *Let  $s$  and  $t \geq (2 \cdot \lceil 8s \log_2 s + 1 \rceil! \cdot (2s)^{\lceil 8s \log_2 s + 1 \rceil})^2$  be positive integers. Let  $G$  be a graph of order  $n$  with  $n \leq 10t/9$  and*

$$e(G) > \frac{1}{2}(t + 3s)(n - s + 1).$$

*Then  $G$  contains a  $K_{s,t}^*$ -minor.*

*Proof.* The edge condition implies both that  $n \geq t + 2s + 1$  and that

$$e(G) > \frac{1}{2}(t + 2s - 3)n,$$

so this theorem is an immediate corollary of Lemma 2.2.7. ■

### 2.2.3 Some Lemmas

**Lemma 2.2.9** *Let  $H$  be a connected graph of order  $n$ . If  $\delta(H) \geq k$ , then there exists a partition  $W_1 \cup \dots \cup W_r$  of  $V(H)$  such that, for every  $i$ ,*

- (a)  $H[\bigcup_{j=1}^i W_j]$  is connected,
- (b)  $|W_i| \leq 3$ , and
- (c)

$$|V(H) - \bigcup_{j=1}^i N[W_j]| \leq n \left( \frac{n-k-1}{n} \right)^i.$$

Furthermore,  $W_1$  can be taken to have size 1.

*Proof.* For  $i = 1$ , we can take  $W_1 = \{v\}$  for any  $v \in V(H)$ . Now suppose the lemma holds for  $i = m - 1$ . Let  $Y_m = \bigcup_{j=1}^{m-1} W_j$  and  $X_m = V(H) - Y_m$ . Then

$$\sum_{x \in X_m} |N[x]| \geq (k+1)|X_m|,$$

and so there is a vertex belonging to the closed neighborhood of at least  $(k+1)|X_m|/n$  vertices of  $X_m$ ; let  $v_m$  be such a vertex with minimum distance to  $Y_m$ .

Since every vertex at distance 3 from  $Y_m$  dominates at least  $k+1$  vertices in  $X_m$ , we can take  $v_m$  to be at distance at most 3 from  $Y_m$ . Then simply take  $W_m$  to be  $v_m$  along with the at most 2 vertices along a shortest path from  $v_m$  to  $Y_m$ .

Now  $Y_m \cup W_m$  dominates all but at most a proportion of  $(n - (k+1))/n$  of the vertices of  $X_m$ , and by induction, at most  $n((n-k-1)/n)^m$  vertices of  $V(H)$ . ■

**Lemma 2.2.10** *Let  $\alpha \geq 2$ . Let  $H$  be a connected graph of order  $n$  with  $\delta(H) \geq k$  and  $n \leq \alpha(k+1)$ . Then there exists a connected dominating set  $A \subseteq V(H)$  such that*

$$|A| \leq 3 \log_{\alpha/(\alpha-1)} n.$$

*Proof.* Let  $W_1 \cup \dots \cup W_r$  be the partition of  $V(H)$  guaranteed by Lemma 2.2.9. Let  $m = \lfloor \log_{\alpha/(\alpha-1)} n \rfloor$  and let  $A_0 = \bigcup_{j=1}^m W_j$ . Then the number of vertices of  $H$  which are not dominated by  $A_0$  is at most

$$n \left(1 - \frac{1}{\alpha}\right)^m = \left(\frac{\alpha}{\alpha-1}\right)^x,$$

where  $x = \log_{\alpha/(\alpha-1)}(n) - m$ .

Since  $\alpha \geq 2$ ,  $\left(\frac{\alpha}{\alpha-1}\right)^x < \alpha \leq 2$ , and so  $A_0$  dominates all but at most 1 vertex of  $H$ . Let this vertex, if it exists, be  $z$ . Since  $H$  is connected,  $z$  has a neighbor  $y$  in  $N[A_0]$ . Then  $A = A_0 \cup \{y\}$  is a connected dominating set in  $H$  and  $|A| = |A_0| + 1 \leq 1 + 3(m-1) + 1 < 3 \log_{\alpha/(\alpha-1)} n$ . ■

Applying Lemma 2.2.10  $s$  times, we have the following corollary.

**Corollary 2.2.11** *Let  $\alpha \geq 2$ . Let  $s, k$ , and  $n$  be positive integers such that  $n \leq \alpha(k+1)$ . Let  $H$  be a  $(3s \log_{\alpha/(\alpha-1)} n)$ -connected graph of order  $n$  with  $\delta(H) \geq k + 3s \log_{\alpha/(\alpha-1)} n$ . Then  $V(H)$  contains  $s$  disjoint subsets  $A_1, \dots, A_s$  such that, for every  $i = 1, \dots, s$ ,*

- (i)  $H[A_i]$  is connected,
- (ii)  $|A_i| \leq 3 \log_{\alpha/(\alpha-1)} n$ , and
- (iii)  $A_i$  dominates  $H - \bigcup_{j=1}^{i-1} A_j$ .

It will be useful to have some statements about the connectivity of  $G$ . In order to do this, we will develop a related concept.

If  $H$  is a graph and  $X \subset V(H)$ , we say that  $X$  is  $k$ -separable if  $N[X] \neq V(H)$  and  $|N(X) - X| \leq k$ .

**Lemma 2.2.12** *Let  $H$  be a graph and  $k$  be a positive integer. If  $X$  is an inclusion-minimal  $k$ -separable set in  $H$  and  $S = N(X) - X$ , then  $H[X \cup S]$  is  $(1 + \lceil k/2 \rceil)$ -connected.*

*Proof.* Assume there is a separating set  $D$  of  $H[X \cup S]$  with  $|D| \leq \lceil k/2 \rceil$ . Let  $H_1$  a the component of  $H[X \cup S] - D$  which has minimum size of intersection

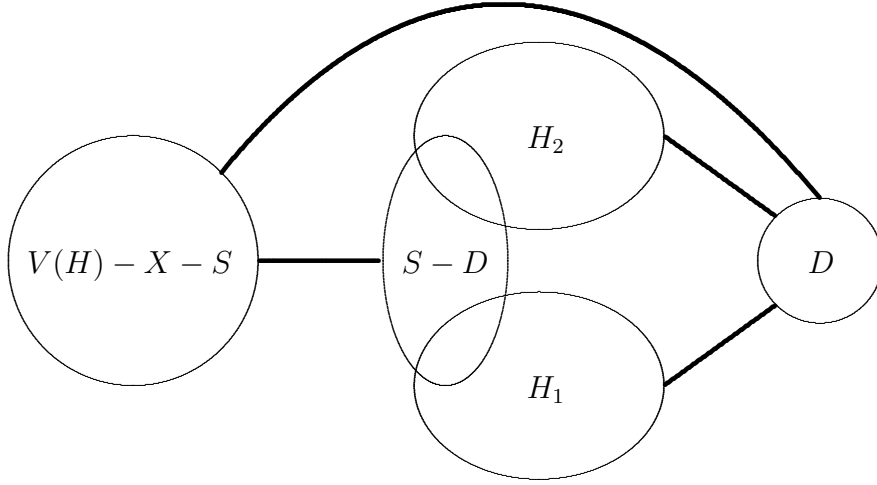


Figure 2.1: Lemma 2.2.12

with  $S$ , and let  $H_2 = H[X \cup S] - D - H_1$ . Then the set  $S_1 = D \cup (S \cap V(H_1))$  has at most  $|D| + |S|/2 \leq k$  vertices and is a separating set of  $H$  (it separates  $H_1 - S_1$  from the rest of the graph; see Figure 2.2.2.1). Moreover,  $H_1 - S_1$  is properly contained in  $X$ , contradicting minimality of  $X$ . Therefore  $H[X \cup S]$  is  $(1 + \lceil k/2 \rceil)$ -connected. ■

Now that we have derived several structural lemmas, we can use them to prove the main result of this subsection, a structural result that will allow us to make reductions on  $G$  in Subsection 2.2.4.

**Lemma 2.2.13** *Let  $G$  be a  $100s \log_2 t$ -connected graph. Suppose that  $G$  contains a vertex subset  $U$  with*

$$t + 100s \log_2 t \leq |U| \leq 3t.$$

*Suppose also that  $\delta(G[U]) \geq 2t/5 + 100s \log_2 t$ . Then  $G$  has a  $K_{s,t}^*$ -minor.*

*Proof.* Perform the following procedure on  $G$ . Let  $i = 1$ . If every component of  $G$  is  $20s \log_2 t$ -connected or  $i > 4$ , stop. Otherwise, let  $S_i$  be the union of a

smallest separating set in  $G[U]$  from each component which was not  $20s \log_2 t$ -connected, and let  $U_i^1, U_i^2, \dots$  be the components of  $G[U] - S_i$ . Increment  $i$  by 1 and iterate.

**Claim 1** *If we did not stop after Step 3, then at most two components of  $G[U] - S_1 - S_2 - S_3 - S_4$  are not  $20s \log_2 t$ -connected.*

*Proof of Claim 1.* Let  $F = G[U] - S_1 - S_2 - S_3 - S_4$ . By construction,  $F$  has at least five components and

$$\delta(F) \geq \delta(G[U]) - 4 \cdot 20s \log_2 t \geq 2t/5 + 20s \log_2 t.$$

Therefore each component of  $F$  has more than  $2t/5 + 20s \log_2 t$  vertices. Furthermore, if any component  $F'$  of  $F$  has fewer than  $4t/5$  vertices, then any two vertices in  $F'$  have at least  $40s \log_2 t$  common neighbors, and so we have  $F'$  is  $40s \log_2 t$ -connected. Hence, if some three components of  $F$  are not  $20s \log_2 t$ -connected, then  $|U| \geq |F| \geq 3 \cdot 4t/5 + 2 \cdot 2t/5 > 3t$ , a contradiction.  $\square$

**Claim 2** *There are an integer  $m$  with  $1 \leq m \leq 3$  and  $m$  vertex disjoint subgraphs  $G_1, \dots, G_m$  of  $G[U]$  such that*

- (1)  $G_i$  is  $20s \log_2 t$ -connected for  $i = 1, \dots, m$ ,
- (2)  $\delta(G_i) \geq 2t/5 + 20s \log_2 t$  for  $i = 1, \dots, m$ , and
- (3)  $|V(G_1)| + \dots + |V(G_m)| \geq t + m20s \log_2 t$ .

*Proof of Claim 2.* Since we deleted at most 4 sets of size at most  $20s \log_2 t$ , we deleted at most  $80s \log_2 t$  neighbors of any vertex, so (2) is immediate.

If we stopped before Step 4, then each component of  $G[U] - S_1 - \dots$  is  $20s \log_2 t$ -connected. By Claim 1, if we stopped after Step 4, then at least three components of  $F$  are  $20s \log_2 t$ -connected (this gives (1)). If we have three such components, then they contain more than  $3 \cdot 2t/5 = 6t/5 > t + 60s \log_2 t$  vertices. If we have at most two such components, then we

stopped before Step 2, so the total number of vertices in them is at least  $|U| - 20s \log_2 t \geq t + 80s \log_2 t$ , which establishes (3).  $\square$

To complete the proof, we consider three cases according to the smallest value of  $m$  for which Claim 2 holds.

**Case 1:**  $m = 1$ . Since  $n(G_1) \leq |U| \leq 3t$ , we know  $n(G_1)/(.4t) \leq 7.5$ . Therefore,

$$3 \log_{7.5/6.5} n(G_1) \leq 3 \log_{7.5/6.5} 3t = \frac{3 \log_2 3t}{\log_2 7.5/6.5} < 20 \log_2 t.$$

Therefore, we can apply Lemma 2.2.11 to  $G_1$  with  $\alpha = 7.5$ . This gives us  $s$  disjoint subsets  $A_1, \dots, A_s$  of  $V(G_1)$  such that, for every  $i = 1, \dots, s$ ,

- (1)  $G[A_i]$  is connected,
- (2)  $|A_i| \leq 3 \log_{7.5/6.5} 3t \leq 20 \log_2 t$ , and
- (3)  $A_i$  dominates  $G_1 - A_1 - \dots - A_{i-1}$ .

Since  $|V(G_1) - A_1 - \dots - A_s| \geq t + 20s \log_2 t - s \cdot 20 \log_2 t = t$ , contracting each  $A_i$  to a single vertex gives us a  $K_{s,t}^*$ -minor in  $G_1$ .

**Case 2:**  $m = 2$ . If either  $G_i$  had order at least  $6t/5$  for  $i \in \{1, 2\}$ , then we could apply the argument of Case 1 to it, so assume otherwise. Then we can apply Lemma 2.2.11 to each of  $G_1$  and  $G_2$  with  $\alpha = (6t/5)/(2t/5) = 3$ . This gives, for  $i = 1, 2$ ,  $s$  disjoint subsets  $A_1^i, \dots, A_s^i$  of  $V(G_i)$  such that, for all  $j \in \{1, \dots, s\}$ ,

- (1)  $G[A_j^i]$  is connected,
- (2)  $|A_j^i| \leq 3 \log_{3/2} 6t/5 < 5.2 \log_2 6t/5 < 7 \log_2 t$ , and
- (3)  $A_j^i$  dominates  $G_i - A_1^i - \dots - A_{j-1}^i$ .

For  $i \in \{1, 2\}$ , define  $A_i = \bigcup_{k=1}^s A_k^i$  and  $V_i = V(G_i) - A_i$ . Since the connectivity of  $G - A_1 - A_2$  is at least  $100s \log_2 t - 14s \log_2 t > s$ , there are  $s$  vertex disjoint  $V_1, V_2$ -paths in  $G - A_1 - A_2$ ; let  $P_1, \dots, P_s$  be a set of such paths such that  $P_q$  contains exactly one vertex in each  $V_i$  for all  $q = 1, \dots, s$ .

For  $j \in \{1, \dots, s\}$ , define  $B_j = A_j^1 \cup A_j^2 \cup V(P_j)$ . Then, by property (3),  $G[B_j]$  is connected for each  $j$  and  $B_j$  dominates both  $V_0 = V_1 \cup V_2 - \bigcup_{q=1}^s V(P_q)$  and  $B_k$  for  $k > j$ . Since we have that

$$\begin{aligned} |V_0| &\geq |V_1 \cup V_2| - 2s \geq |V_1| + |V_2| - |A_1| - |A_2| - 2s \\ &\geq t + 40s \log_2 t - 14s \log_2 t - 2s > t, \end{aligned}$$

we can safely contract each  $B_j$  to a single vertex to obtain a  $K_{s,t}^*$ -minor.

**Case 3:**  $m = 3$ . Again we may assume that Cases 1 and 2 do not hold. If  $n(G_1) \geq 4t/5$ , then  $n(G_1) + n(G_2) > 4t/5 + \delta(G_2) > 6t/5$ , and Case 2 would apply. Therefore, invoking symmetry, we can say that  $n(G_i) \leq 4t/5$  for  $i \in \{1, 2, 3\}$ .

Apply Lemma 2.2.11 to each of  $G_1$ ,  $G_2$ , and  $G_3$  with  $\alpha = 2$ . In this case, the Lemma guarantees us, for each  $i \in \{1, 2, 3\}$ , disjoint subsets  $A_1^i, \dots, A_s^i$  of  $V(G_i)$  such that, for every  $j \in \{1, \dots, s\}$

- (1)  $G[A_j^i]$  is connected,
- (2)  $|A_j^i| \leq 3 \log_2 4t/5 < 3 \log_2 t$ , and
- (3)  $A_j^i$  dominates  $G_i - A_1^i - \dots - A_{j-1}^i$ .

Define  $V_i = V(G_i) - \bigcup_{j=1}^s A_j^i$  for all  $i$ . Then

$$|V_1 \cup V_2 \cup V_3| \geq 3(2t/5 + 20s \log_2 t) - 3s \cdot 3 \log_2 t = 6t/5 + 51s \log_2 t.$$

For all  $i$ , choose  $X_i \subset V_i$  such that  $|X_1| = 2s$  and  $|X_2| = |X_3| = s$ . Since the connectivity of the graph  $G_0 = G - \bigcup_{i=1}^3 \bigcup_{j=1}^s A_j^i$  is at least  $100s \log_2 t - 9s \log_2 t = 91s \log_2 t$ , there are  $2s$  vertex-disjoint  $(X_1, X_2 \cup X_3)$ -paths  $P_1, \dots, P_{2s}$  in  $G_0$ . Renumber these paths so that  $P_q$  is an  $(X_1, X_2)$ -path if  $q$  is odd (and consequently  $P_q$  is an  $(X_1, X_3)$ -path if  $q$  is even). Then we can find (see Figure 2.2.2.2) paths  $Q_1, \dots, Q_{2s}$  such that  $Q_q$  is a subpath of  $P_q$  and, for every  $k \in \{1, \dots, s\}$ ,

- (a)  $Q_{2k-1} \cup Q_{2k} \subseteq P_{2k-1} \cup P_{2k}$ ,

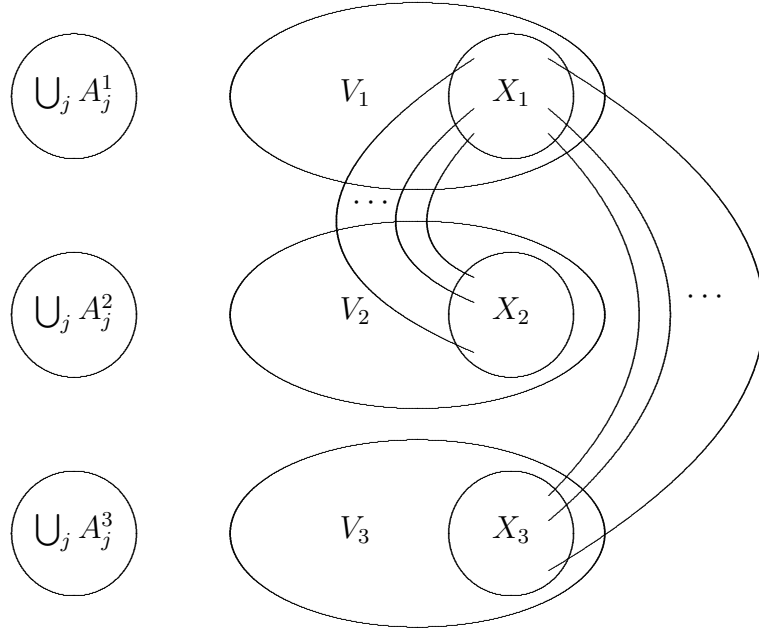


Figure 2.2: The paths  $Q_q$

- (b)  $|V(Q_{2k-1} \cup Q_{2k}) \cap (V_1 \cup V_2 \cup V_3)| \leq 4$ , and  
(c)  $V(Q_{2k-1} \cup Q_{2k}) \cap U_i \neq \emptyset$  for all  $i$ .

For  $k \in \{1, \dots, s\}$ , let  $W_k = Q_{2k-1} \cup Q_{2k} \cup A_k^1 \cup A_k^2 \cup A_k^3$ . Then we know

- (1)  $G[W_k]$  is connected for all  $k$ ,
- (2) the  $W_k$ 's are pairwise vertex disjoint, and
- (3)  $W_k$  dominates  $V_1 \cup V_2 \cup V_3 - \bigcup_{q=1}^{2s} Q_q$  and all  $W_j$  with  $j > k$ .

Since  $|V_1 \cup V_2 \cup V_3 - \bigcup_{q=1}^{2s} Q_q| \geq 6t/5 + 91s \log_2 t - 4s > t$ ,  $G$  has a  $K_{s,t}^*$ -minor, and the Lemma is proved.  $\blacksquare$

## 2.2.4 The Final Argument

We are finally ready to prove the main result of this section, namely Theorem 2.1.4.

For a set  $W \subseteq V(G)$ , define  $d_G(W)$  to be the number of edges that intersect  $W$ .

**Theorem 2.2.14** *Let  $s \geq 3$  and  $t \geq 2 \cdot 10^5 \cdot s^2$  be integers. Let  $G$  be a graph of order  $n \geq 10t/9$  satisfying*

- (1) *for every  $W \subseteq V(G)$ ,  $d_G(W) \geq |W|t/2$ ,*
- (2) *every edge of  $G$  is contained in at least  $t/2$  triangles,*
- (3)  *$e(G) \geq tn/2$ , and*
- (4) *no subgraph of  $G$  has average degree at least  $11t/10$ .*

*Then  $G$  contains a  $K_{s,t}^*$ -minor.*

*Proof.* We consider cases for the connectivity of  $G$ .

**Case 1:**  $G$  is  $200s \log_2 t$ -connected. If  $G$  has a vertex  $v$  satisfying

$$t + 100s \log_2 t \leq d_G(v) \leq 3t - 1,$$

then  $G$  satisfies Lemma 2.2.13 with  $U = N[v]$  (we know  $\delta(G[N[v]]) \geq t/2 \geq 2t/5 + 100s \log_2(t)$  since there are  $t/2$  triangles on every edge of  $G$ ), so assume no such vertex exists.

Let  $V_{\text{sm}}$  be the set of vertices in  $G$  with degree less than  $t + 100s \log_2 t$ . If  $|V_{\text{sm}}| > t + 100s \log_2 t$ , then there is a set  $V'_{\text{sm}} \subseteq V_{\text{sm}}$  such that

$$t + 100s \log_2 t \leq \left| \bigcup_{v \in V'_{\text{sm}}} N_G[v] \right| \leq 3t - 1$$

(we can find such a set by successively adding the closed neighborhood of a single vertex in  $V_{\text{sm}} - V'_{\text{sm}}$  to  $V'_{\text{sm}}$ ). Then we can apply Lemma 2.2.13 with  $U = N[V'_{\text{sm}}]$  to find our minor.

Thus we have that  $|V'_{\text{sm}}| \leq t + 100s \log_2 t$ . Since every vertex not in  $V'_{\text{sm}}$  has degree at least  $3t$ , we can see that

$$\frac{t|V_{\text{sm}}|}{2} + 3t(n - |V_{\text{sm}}|) \leq \sum_{v \in V(G)} d_G(v) < 11tn/10$$

where the inequality on the right follows from (4). This tells us that

$$n < \frac{2.5|V_{\text{sm}}|}{1.9} < 2|V_{\text{sm}}| < 3t.$$

If  $n > t + 100s \log_2 t$ , then apply Lemma 2.2.13 with  $U = V(G)$ . Otherwise, we are done by applying Lemma 2.2.8.

**Case 2:**  $G$  is not  $200s \log_2 t$ -connected. Let  $k = \lceil 200s \log_2 t \rceil$  and let  $S$  be a separating set in  $G$  of size less than  $k$ . Let  $V_1$  and  $V_2$  be nonempty and partition  $V(G) - S$  such that  $E_{G-S}[V_1, V_2] = \emptyset$ . Since  $N(V_i) - V_i \subseteq S$ , we know  $|N(V_i) - V_i| \leq |S| < k$ , so  $V_i$  is  $k$ -separable for  $i = 1, 2$ . Let  $W_i$  be an inclusion-minimal  $k$ -separable set contained in  $V_i$  and let  $S_i = N(W_i) - W_i$ . Then we have, by Lemma 2.2.12, that the graph  $G_i = G[W_i \cup S_i]$  is  $100s \log_2 t$ -connected.

**Case 2.1:**  $|W_i \cup S_i| \geq t + 100s \log_2 t$  for some  $i$ . In this case, we essentially recreate the argument for Case 1, with only some slight modifications.

Fix  $i$  such that the condition of this Case is satisfied. Then  $G_i$  has no vertex  $v$  with

$$t + 100s \log_2 t \leq d_{G_i}(v) \leq 3t - 1,$$

by Lemma 2.2.13. Let  $V_{\text{sm}} = \{v \in G_i : d_{G_i}(v) < t + 100s \log_2 t\}$ . If  $V_{\text{sm}}$  has size larger than  $t + 100s \log_2 t$ , then again we are done by Lemma 2.2.13. Otherwise,  $|V_{\text{sm}}| \leq t + 100s \log_2 t$ , so

$$n(G_i) < \frac{2.5|V_{\text{sm}}|}{1.9} < 3t.$$

Apply Lemma 2.2.13 to  $G_i$  with  $U = V(G_i)$ .

**Case 2.2:**  $|W_i \cup S_i| < t + 100s \log_2 t$  for  $i = 1, 2$ . Each vertex in  $W_i$  is adjacent to at most  $|S_i| \leq 200s \log_2 t$  vertices outside  $W_i$ . Therefore at most  $|W_i|^2 + 200s \log_2 t |W_i|$  edges are incident to  $W_i$ . On the other hand,

by condition (1), we know  $W_i$  must be incident to at least  $t|W_i|/2$  edges. Putting these together, we get  $|W_i| \geq t - 400s \log_2 t$ .

If we define  $H_i$  to be  $G[W_i]$ , then clearly

$$\delta(H_i) \geq \delta(G) - |S_i| \geq t/2 - 200s \log_2 t.$$

Also,  $H_i$  must be  $400s \log_2 t$ -connected. To see this, suppose  $S_0$  were a separating set in  $H_i$  of size less than  $400s \log_2 t$ . Let  $H_0$  be the smallest component of  $H_i - S_0$ , so that  $H_0$  has order at most  $t/2 + 50s \log_2 t$ . Then the number of edges of  $G$  incident to  $H_0$  is at most  $|H_0|^2/2 + |S_i||H_0|$ . Again, though, by (1), there must be at least  $t|H_0|/2$  edges incident to  $H_0$ . However,

$$\frac{|H_0|^2}{2} + |S_i||H_0| \leq \frac{|H_0|^2}{2} + |H_0|200s \log_2 t \leq |H_0| \left( \frac{t}{4} + 225s \log_2 t \right) < \frac{t|H_0|}{2},$$

a contradiction. Therefore  $\kappa(H_i) \geq 400s \log_2 t$ .

In summary, we have

- (a)  $n(H_i) \leq t + 100s \log_2 t \leq 6t/5$ ,
- (b)  $\kappa(H_i) \geq 400s \log_2 t \geq 3s \log_{3/2} n(H_i)$ , and
- (c)  $\delta(H_i) \geq t/2 - 200s \log_2 t \geq 2t/5 + 3s \log_{3/2} n(H_i)$ .

We may therefore apply Lemma 2.2.11 to  $H_i$  (with  $k = 2t/5$  and  $\alpha = 3$ ) to find  $s$  disjoint subsets  $A_1^i, \dots, A_s^i$  of  $V(H_i)$  such that, for  $j \in \{1, \dots, s\}$ ,

- (1)  $H_i[A_j^i]$  is connected,
- (2)  $|A_j^i| \leq 3 \log_{3/2} |W_i| < 6 \log_2 |W_i|$ , and
- (3)  $A_j^i$  dominates  $W_i - A_1^i - \dots - A_{j-1}^i$ .

Since  $G$  is  $s$ -connected, there are  $s$  pairwise vertex disjoint  $S_1, S_2$ -paths  $P_1, \dots, P_s$ . We may assume that  $S_i$  and  $P_j$  only intersect in one vertex; call this vertex  $p_{i,j}$ . Since there are at least  $t/2$  triangles on every edge of  $G$ , and since  $p_{i,j}$  has a neighbor  $w_{i,j} \in W_i$ , we know  $p_{i,j}$  must have at least  $t/2 - |S_i| \geq t/2 - 200s \log_2 t$  neighbors in  $W_i$ . Therefore, we can find, in a greedy fashion,  $2s$  distinct vertices

$$q_{i,j} \in W_i - \bigcup_{k=1}^s A_k^i$$

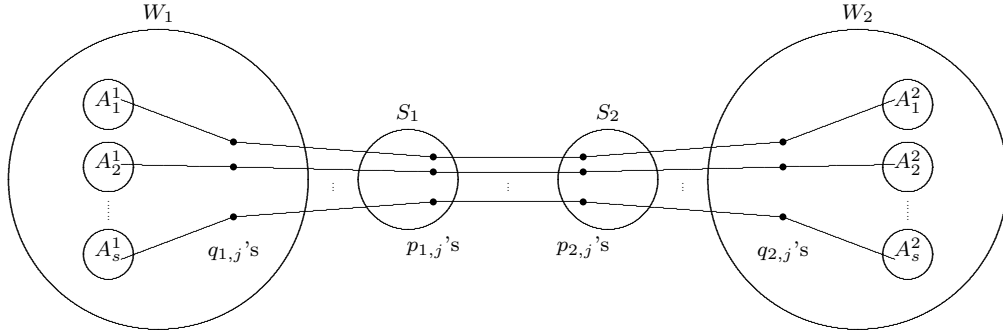


Figure 2.3: The sets  $F_j$

such that  $p_{i,j}q_{i,j} \in E(G)$ .

Define  $F_j = A_j^1 \cup A_j^2 \cup V(P_j) \cup \{q_{1,j}, q_{2,j}\}$  for  $j \in \{1, \dots, s\}$ . Then we know, for all  $j$ ,

- (i)  $G[F_j]$  is connected,
- (ii) the  $F_j$ 's are pairwise disjoint, and
- (iii)  $F_j$  dominates  $\bigcup_{i=1}^2 W_i - F_1 - \dots - F_{j-1}$ .

Since

$$\left| \bigcup_{i=1}^2 W_i - F_1 - \dots - F_{j-1} \right| \geq 2(t - 400s \log_2 t - 6s \log_2 2t - s) > t,$$

contracting each  $F_j$  to a single vertex shows that  $G$  has a  $K_{s,t}^*$ -minor.

This completes the proof. ■

**Theorem 2.2.15** *Let  $s \geq 3$  and  $t > (2 \cdot \lceil 8s \log_2 s + 1 \rceil! \cdot (2s)^{\lceil 8s \log_2 s + 1 \rceil})^2$  be integers. Let  $G$  be a graph of order  $n$  with*

$$e(G) > \frac{t + 3s}{2}(n - s + 1).$$

*Then  $G$  contains a  $K_{s,t}^*$ -minor.*

*Proof.* Let  $G$  be as above, and let  $G'$  be a subgraph of  $G$  which is minor-minimal with respect to the edge condition.

If  $n(G') < 10t/9$ , we are done by Lemma 2.2.8, so suppose that  $n(G') \geq 10t/9$ .

If any subset  $W$  of  $V(G')$  were adjacent to fewer than  $|W|t/2$  edges,  $G' - W$  would satisfy the edge condition. Similarly, if there were an edge of  $G'$  in fewer than  $t/2$  triangles, then contracting this edge would give a minor of  $G'$  satisfying the edge condition. Finally, if any subgraph of  $G'$  had average degree at least  $11t/10$ , it too would satisfy the edge condition. All these contradict the choice of  $G'$ , so none hold. Thus we can apply Theorem 2.2.14 to  $G'$  to find a  $K_{s,t}^*$ -minor in  $G'$  and hence in  $G$ . ■

### 2.3 The Case $s = 3$

For small values of  $s$ , Myers' construction is optimal: Myers [41] and Chudnovsky, Reed, and Seymour [18] showed this for  $s = 2$ . The aim of this section is to show the same for  $s = 3$  by proving Theorem 2.1.6. The most difficult case, when  $G$  has order at most  $t + 6$ , will be handled by Lemma 2.3.3. After that we will use a  $\Delta$ -system argument to reduce the problem to graphs with at least  $t + 33$  vertices, at which point we can use results of Subsection 2.2.2 to complete the proof.

**Proposition 2.3.1** *Let  $G$  be a graph of order  $n$  with  $e(G) > \frac{1}{2}(t + 3)(n - 2) + 1$ . If  $G$  has a dominating vertex, then  $G$  contains a  $K_{3,t}$ -minor.*

*Proof.* If  $v$  is a dominating vertex in  $G$ , then

$$e(G - v) > \frac{1}{2}(t + 3)(n - 2) + 1 - (n - 1) = \frac{1}{2}(t + 1)(n(G - v) - 1),$$

so by Theorem 2.1.2,  $G - v$  has a  $K_{2,t}$ -minor. Adding  $v$  to the smaller partite set gives a  $K_{3,t}$ -minor in  $G$ . ■

**Lemma 2.3.2** *Let  $G$  be a graph of order  $n = t + 5 \geq 108900$  which is minor-minimal with respect to the condition*

$$e(G) > \frac{1}{2}(t + 3)(n - 2) + 1.$$

*Then  $G$  contains a  $K_{3,t}$ -minor.*

*Proof.* We will prove this using a discharging argument. Let  $H = \overline{G}$ . We begin with several structural claims about  $H$ . Note that the edge condition on  $G$  tells us that  $e(H) < (3n - 7)/2$ . Also,  $\delta(G) \geq (t + 3)/2$ , otherwise  $G$  would not be subgraph-minimal with respect to the edge condition. Finally,  $H$  has no isolated vertices by Proposition 2.3.1.

**Claim 1**  *$H$  is connected.*

*Proof of Claim.* Suppose  $H$  is disconnected.  $H$  has no isolated vertices, so every component of  $H$  has at least two vertices. If  $H$  has a component of order 2, we can delete the neighbors of a vertex of degree at most 2 in  $H$  to get a  $K_{3,t}$ -subgraph in  $G$ . Otherwise, let  $v$  be a vertex of degree at most 2 in  $H$ . We can find two disjoint pairs  $(u_1, w_1)$  and  $(u_2, w_2)$  such that  $u_i$  and  $w_i$  are in different components of  $H$  and  $N_H(v) \subseteq \{u_1, u_2\}$ . In  $G$ , contract both edges  $u_1w_1$  and  $u_2w_2$  to get a  $K_{3,t}$ -minor.  $\square$

**Claim 2** *If there is a vertex  $x$  adjacent to two vertices of degree 1 in  $H$ , then  $H$  has no other vertices of degree 1.*

*Proof of Claim.* Suppose there is another vertex of degree 1 with neighbor  $y$ . Then  $G - x - y$  has three dominating vertices, and thus  $G$  has a  $K_{3,t}$ -subgraph.  $\square$

**Claim 3** *For every  $x, y \in V(H)$ , the number of components of order at most 3 in  $H - x - y$  is at most 2.*

*Proof of Claim.* Suppose to the contrary that  $H - x - y$  contains three components  $C_1, C_2,$  and  $C_3$  of order at most 3. Then  $|V(C_i)| = 2$  for each  $i$ , otherwise  $G - x - y$  would have a  $K_{3,t}$ -subgraph. Let  $N_{x,y} = N_H(x) \cap N_H(y)$ , and define  $N_x = N_H(x) - N_{x,y}$ ,  $N_y = N_H(y) - N_{x,y}$ , and  $R = V(G) - N_x - N_y - N_{x,y}$ .

Suppose there is a vertex  $z \in R$  with no neighbor in  $N_{x,y}$  (in  $H$ ). In  $G$ , contract the edges  $xz$  and  $yz$  to get a dominating vertex, which together with the two vertices in some  $C_i$  not containing  $z$  gives the smaller partite set of a  $K_{3,t}$ -minor. We conclude that for every  $z \in R$  there is a common neighbor in  $H$  of  $x, y,$  and  $z$ .

Now suppose that the only common neighbor (if one exists) of  $x$  and  $z \in N_x$  is  $y$ . In  $G$ , delete  $y$  and contract  $xz$  to get a new vertex  $v$ . Then  $v$  dominates  $G - y$ , so it along with the two vertices of some  $C_i$  not containing  $z$  form the smaller partite set of a  $K_{3,t}$ -minor. We conclude that  $z$  and  $x$  have a common neighbor other than  $y$ , and by symmetry  $y$  and every  $z \in N_y$  have a common neighbor other than  $x$ .

We now aim to show that  $R$  contains two vertices of degree 1 with a common neighbor in  $N_{x,y}$ . To see this, assume otherwise, and consider the following discharging scheme from  $E(H)$  to  $V(H)$ . Each edge initially carries weight 1. Every edge of  $E_H(\{x, y\}, N_x \cup N_y)$  gives its weight to the endpoint in  $N_x \cup N_y$ . If an edge connects  $\{x, y\}$  to  $z \in N_{x,y}$  and  $z$  has a neighbor  $w$  of degree 1 in  $R$ , it gives weight  $3/4$  to  $z$  and  $1/4$  to  $w$ . Otherwise, this edge gives all its weight to  $z$ . Any edge of  $E_H(R, V(H) - R)$  gives all its weight to its endpoint in  $R$ , and all other edges give weight  $1/2$  to each endpoint. As a result of this discharging, each vertex in  $N_{x,y}$  receives at least  $2 \cdot (3/4)$  from the edges connecting it to  $\{x, y\}$ . Every vertex in  $N_x \cup N_y$  receives 1 from the edge connecting it to  $\{x, y\}$  and at least  $1/2$  from the edges in  $N_x \cup N_y \cup N_{x,y}$  (from the edge connecting it to the common neighbor guaranteed by the third paragraph). Each vertex  $z \in R$  receives at least 1 from a neighbor  $z' \in N_{x,y}$  ( $z'$  being the common neighbor of  $x, y,$  and  $z$

guaranteed by the second paragraph), and another  $1/2$  from a neighbor in  $N_x \cup N_y \cup N_{x,y}$  (if it has degree greater than 1) or an additional  $2 \cdot (1/4)$  from the edges connecting  $z$  to  $\{x, y\}$ . Therefore, every vertex outside of  $\{x, y\}$  ends up with weight at least  $3/2$ , so  $e(H) \geq 3(n-2)/2$ , a contradiction.

Let  $z \in N_{x,y}$  be the vertex with two neighbors in  $R$  of degree 1 (whose existence is guaranteed by the last paragraph), and let  $w_1, w_2 \in R$  be neighbors of  $z$  with degree 1 in  $H$ . Note that the discharging of the last paragraph gives all vertices charge  $3/2$  except  $x, y$ , and  $z$  (which gets  $1/2$ ). If  $xy \in E(H)$  or some  $C_i$  sends at least three edges to  $\{x, y\}$ , then the sum of the terminal charges is still at least  $3(n-2)/2$ , so neither of these happens. If any vertex  $a$  of  $C_i$  had degree 1 in  $H$  and  $j \neq i$ , then delete  $y$  and contract  $ax$  in  $G$ , and this new vertex along with the vertices of  $C_j$  represent the smaller partite set of a  $K_{3,t}$ -minor. Therefore each vertex of each  $C_i$  has a neighbor in  $\{x, y\}$ . Let  $V(C_1) = \{v_1, v_2\}$ . If both  $v_1$  and  $v_2$  are adjacent to  $x$ , then in  $H - x - z$  the vertices  $v_1, v_2$ , and  $w_1$  have no neighbors outside of themselves, so in  $G$  they are the smaller partite set of a  $K_{3,t}$ -subgraph. We may therefore assume that  $xv_1, yv_2 \in E(H)$ . In  $G$ , contract  $v_1y$  and  $v_1z$ . Then  $\{w_1, w_2, v_2\}$  represent the smaller partite set of a  $K_{3,t}$ -minor in  $G$ .  $\square$

**Claim 4** *For every  $x, y \in V(H)$ , the number of tree-components in  $H - x - y$  that send at most two edges to  $\{x, y\}$  is at most 3.*

*Proof of Claim.* Suppose there are  $x, y \in V(H)$  such that  $H - x - y$  has at least four tree-components which send at most two edges to  $\{x, y\}$ ; let  $C_1, C_2, C_3$ , and  $C_4$  be four such tree-components. We can assume without loss of generality, by Claim 3, that  $|V(C_1)| \geq |V(C_2)| \geq 4$ .

Suppose that  $x$  has no neighbor in  $C_i$  for some  $i \in \{1, 2\}$ . Let  $v_1, \dots, v_k$  be the vertices of a longest path in  $C_i$ . If there is a leaf adjacent to  $v_2$  other than  $v_1$ , then in  $G$  we can contract  $v_2$  into a vertex of  $C_{3-i}$  and delete  $y$ . The resulting graph has three dominating vertices (the leaves adjacent to  $v_2$

and the new vertex), so  $G$  has a  $K_{3,t}$ -minor. If  $v_2$  has no leaf neighbor other than  $v_1$ , then it must have degree 2. In  $G$ , contract  $v_3$  into a vertex of  $C_{3-i}$  and delete  $y$ . Then  $v_1, v_2$ , and the new vertex are the smaller partite set of a  $K_{3,t}$ -minor. We may therefore assume, by symmetry, that both  $x$  and  $y$  have a neighbor in  $C_1$  and  $C_2$ .

Now suppose that  $x$  and  $y$  have a common neighbor  $z$  in  $C_i$  for some  $i \in \{1, 2\}$ . Consider  $C_i - z$ . If it has a component with at least 4 vertices, then it has a longest path  $v_1, \dots, v_k$  such that neither  $v_1$  nor  $v_k$  is adjacent to  $z$  and such that if  $v_2$  is adjacent to only one leaf neighbor in  $C_i - z$ , then  $v_2$  has degree 2 in  $H$ . Applying the argument of the last paragraph shows that  $G$  has a  $K_{3,t}$ -minor. We may therefore assume that each component of  $C_i - z$  has order at most 3. If any component of  $C_i - z$  has order exactly 3, then the vertices of this component form the smaller partite set of a  $K_{3,t}$ -subgraph in  $G$ . If a component of  $C_i - z$  has order 1, then since  $C_i - z$  has at least three vertices, there is a set of components of  $C_i - z$  whose union has size exactly 3, so again  $G - z$  has a  $K_{3,t}$ -subgraph. Therefore each component of  $C_i - z$  has order exactly 2. Let  $D_1$  and  $D_2$  be two such components and let  $d$  be a vertex of  $D_1$ . Then  $D_1 \cup D_2 - d$  forms the smaller partite set of a  $K_{3,t}$ -subgraph in  $G - z - d$ . We may assume, therefore, that  $x$  and  $y$  have no common neighbor in either  $C_i$ .

By the arguments of the last two paragraphs, for  $i \in \{1, 2\}$ , there is a vertex  $v_{i,x} \in (N_H(x) - N_H(y)) \cap V(C_i)$  and a  $v_{i,y} \in (N_H(y) - N_H(x)) \cap V(C_i)$ .

Suppose that  $v_{i,x}$  is a separating vertex of  $C_i$ , and let  $D$  be the set of vertices of all components of  $C_i - v_{i,x}$  which do not contain  $v_{i,y}$ . Contract the edge  $v_{i,x}v_{3-i,y}$  in  $G$  to get a new vertex  $v$ . If  $|D|$  is 2 or 3 or if  $C_i - v_{i,x}$  has two isolated vertices, then  $D \cup \{v\}$  contains the smaller partite set of a  $K_{3,t}$ -minor in  $G$ . If  $|D| = 1$ , then let  $\ell$  be a leaf in the component of  $C_i - v_{i,x}$  containing  $v_{i,y}$  such that  $\ell \neq v_{i,y}$ . Deleting the neighbor in  $H$  of  $\ell$ , we get a  $K_{3,t}$ -minor of  $G$  whose smaller partite set is  $D \cup \{\ell, v\}$ . We are left only with the case that  $|D| \geq 4$  and  $D$  has a component  $C^*$  with more than one

vertex. Contract the neighbor of a leaf  $\ell$  in  $D$  with a vertex in  $C_{3-i}$ . This new vertex along with  $\ell$  and  $v$  forms the smaller partite set of a  $K_{3,t}$ -minor in  $G$ . We conclude that  $v_{i,x}$  and  $v_{i,y}$  are leaves in  $C_i$ .

Let  $v_1, \dots, v_k$  be the path in  $C_1$  from  $v_1 = v_{1,x}$  to  $v_k = v_{1,y}$ . We know  $k \geq 3$  since  $|V(C_1)| > 2$ . In  $G$ , contract  $v_1 v_{2,y}$  to get a new dominating vertex  $v$ . If there is a leaf  $\ell \in C_1 - \{v_1, v_k\}$ , contract its neighbor into  $v_{2,x}$  in  $G$  to get a new vertex  $w$ ; now  $\{v, w, \ell\}$  is the smaller partite set of a  $K_{3,t}$ -minor. Otherwise, let  $w$  be the vertex resulting from contracting  $v_3 v_{2,x}$  in  $G$ ; now  $\{v, w, v_2\}$  is the smaller partite set of a  $K_{3,t}$ -minor.

This completes the proof of the Claim.  $\square$

**Claim 5**  $H$  has no dominating set of size less than  $\sqrt{n}/10$ .

*Proof of Claim.* Let  $S$  be a dominating set of  $H$  with  $|S| = s < \sqrt{n}/10$ . Let  $T = V(H) - S$ . Let  $H_T = H[T]$ . Let  $m$  be the number of components of  $H_T$  which are trees. Since  $S$  dominates  $H$ ,  $|E_H(S, T)| \geq |T|$ . Therefore

$$e(H_T) < (3n - 7)/2 - |T| < n/2 - 3 + s,$$

so  $m \geq n/2 + 3 - 2s$ . Let  $c_{i,j}$  denote the number of tree components  $C$  of  $H_T$  with  $i$  vertices and  $|E_H(C, S)| = j$ .

Claim 4 tells us that  $\sum_i (c_{i,1} + c_{i,2}) \leq 3\binom{s}{2}$ . Also,  $e(H_T) \geq n(H_T) - m = n - s - m$ . Therefore

$$e(H) \geq e(H_T) + |E_H(S, T)| \quad (2.5)$$

$$= e(H_T) + \sum_i \sum_j j \cdot c_{i,j} \quad (2.6)$$

$$\geq (n - s - m) + 3m - \sum_i (2c_{i,1} + c_{i,2}) \quad (2.7)$$

$$\geq n - s + 2m - 2 \sum_i (c_{i,1} + c_{i,2}) \quad (2.8)$$

$$\geq n - s + 2m - 6 \binom{s}{2} \quad (2.9)$$

$$\geq n - s + 2(n/2 + 3 - 2s) - 6 \binom{s}{2} \quad (2.10)$$

$$\geq 2n + 6 - 5s - \frac{3n}{100} \quad (2.11)$$

$$> 3n/2 \quad (2.12)$$

$$(2.13)$$

since  $n \geq 5$ . This contradiction finishes the proof of the Claim.  $\square$

**Claim 6** *Every vertex of degree 2 in  $H$  has a neighbor of degree at least  $\sqrt{n}/10 - 10$ .*

*Proof of Claim.* Let  $N_H(v) = \{u, w\}$  and suppose  $u$  and  $w$  have degree at most  $\sqrt{n}/10 - 10$ . By Claim 5, there must be at least 8 vertices at distance at least 3 from  $x$ ; otherwise,  $N_H[u] \cup N_H(v)$  along with the vertices at distance at least 3 from  $x$  constitute a dominating set of  $H$  with size at most  $\sqrt{n}/10$ . The same holds true with  $y$  in place of  $x$ , so we can find distinct  $x'$  and  $y'$  at distance at least 3 from  $x$  and  $y$ , respectively. In  $G$ , contract  $xx'$  and  $yy'$ . These two new vertices along with  $v$  form the smaller partite set of a  $K_{3,t}$ -minor.  $\square$

Define a *2-vertex* to be a vertex of degree 2 in  $H$ , and say that a 2-vertex is *weak* if one of its neighbors has degree at most 5.

**Claim 7** *If a vertex  $v \in V(H)$  is adjacent to at least 15 weak 2-vertices, then it has at least 5 neighbors of degree at least 3.*

*Proof of Claim.* Suppose otherwise, and let the weak 2-vertices adjacent to  $v$  be  $z_1, \dots, z_s$  with  $s \geq 15$ . Let  $y_i$  be the neighbor of  $z_i$  which is not  $v$ . Let  $x_1, \dots, x_k$  be the neighbors of  $v$  with degree at least 4 (so our assumption implies  $k \leq 4$ ). The set  $\{y_j : 1 \leq j \leq s\}$  has cardinality at least 8, otherwise some three 2-vertices have the same neighborhood  $N$ , implying that  $G - N$  has a  $K_{3,t}^*$ -subgraph. Let  $X = \{x_1, \dots, x_k\}$  and  $Y = \{y_1, \dots, y_s\}$ .

Suppose some  $y_j \in Y - X$  has no neighbor in  $X$ . Let  $z_j$  be a neighbor of  $y_j$ . Contract  $vy_j$  in  $G$  to get a new vertex  $v'$ . If  $v$  and  $y_j$  have one common neighbor in  $H$  and  $z_{j'}$  is not adjacent to  $y_j$ , then delete the neighbor in  $H$  of  $z_{j'}$ ; now  $\{v', z_j, z_{j'}\}$  is the smaller partite set of a  $K_{3,t}$ -minor in  $G$ . If  $v$  and  $y_j$  had two common neighbors, then these two along with  $v'$  are the smaller partite set of the  $K_{3,t}$ -minor. Therefore every  $y_j \in Y - X$  has a neighbor in  $X$ .

Suppose now that there is in  $H$  a vertex  $w$  at distance at least 3 from  $v$ . Since  $z_1, \dots, z_s$  are weak, each  $y_j$  has at most 4 neighbors in  $Y$ . We can assume without loss of generality that  $y_1y_2 \notin E(H)$ . In  $G$ , contract  $y_1y_2$  and  $vw$ . Together with  $z_1$  and  $z_2$ , the vertex created by contracting  $vw$  form the smaller partite set of a  $K_{3,t}$ -minor in  $G$ .

Putting the last two paragraphs together,  $X \cup \{v\}$  is a dominating set in  $H$ , contradicting Claim 5.  $\square$

**Claim 8** *If a vertex  $v \in V(H)$  with degree at most  $\sqrt{n}/10 - 10$  is adjacent to a vertex of degree 1, then there are no other vertices of degree 1 in  $H$ .*

*Proof of Claim.* Let  $y$  be a neighbor of  $v$  with degree 1. Let  $z$  be another vertex of degree 1 whose neighbor is  $z'$ . By Claim 5 there are at least 9

vertices at distance at least 3 from  $v$  (otherwise these vertices along with  $N_H[v]$  is a dominating set with too small a size). Let  $w$  be such a vertex. In  $G$ , contract  $vw$ . The new vertex along with  $y$  and  $z$  form the smaller partite set of a  $K_{3,t}$ -minor in  $G$ .  $\square$

Consider the following discharging scheme on  $V(H)$ . Initially each vertex has charge  $\phi(v) = d_H(v)$ , so  $\sum_{v \in V(H)} \phi(v) \leq 3n - 7$ . The rules for discharging are as follows:

**Rule 1:** If  $d_H(v) \geq \sqrt{n}/10 - 10$  and  $v$  has neighbors of degree 1, then  $v$  gives charge 2 to one of them and nothing to the rest.

**Rule 2:** If  $v$  is a weak 2-vertex, then it receives charge 1 from the neighbor of the larger degree.

**Rule 3:** If  $v$  is a non-weak 2-vertex, then  $v$  gets charge  $1/2$  from each neighbor.

We will now show that the new charge  $\phi'(v)$  is at least 3 for every  $v$ , with the possible exception of a single vertex. This implies

$$2e(H) = \sum_{v \in V(H)} \phi'(v) \geq 3n - 3,$$

a contradiction. To do this, we consider cases for  $d_H(v)$ .

**Case 1:**  $d(v) = 1$ . If  $\phi'(v) < 3$ ,  $v$  must either have been adjacent to a vertex of degree at least  $\sqrt{n}/10 - 10$  which itself was adjacent to another vertex of degree 1 or  $v$  was adjacent to a vertex of degree at most  $\sqrt{n}/10 - 10$ . In the former case, by Claim 2 there is at most one such vertex. In the latter, by Claim 8, there are no other vertices of degree 1 in  $H$ .

**Case 2:**  $d(v) = 2$ . Since  $v$  never gives away any charge, and Rules 2 and 3 give  $v$  an additional charge of 1,  $\phi'(v) = 3$ .

**Case 3:**  $3 \leq d(v) \leq 5$ . No charge is given away by  $v$ .

**Case 4:**  $6 \leq d(v) < \sqrt{n}/10 - 10$ . Claim 6 tells us every 2-vertex has a neighbor of degree at least  $\sqrt{n}/10 - 10$ , so  $v$  is not adjacent to any weak 2-vertices. Therefore  $v$  only gives charge away through Rule 3, and then at most  $1/2$  to each neighbor. Since it loses at most  $d(v)/2$ , it retains charge at least  $d(v)/2 \geq 3$ .

**Case 5:**  $d(v) \geq \sqrt{n}/10 - 10$ . If  $v$  is adjacent to fewer than 15 weak 2-vertices, then it gives away charge at most  $2 + (d_H(v) - 1)/2 + 14/2$ , and so  $\phi'(v) \geq (d_H(v) - 17)/2 \geq 3$ . If  $v$  is adjacent to at least 15 weak 2-vertices, then by Claim 7,  $v$  gives away at most  $2 + (d_H(v) - 5) = d_H(v) - 3$ , so  $\phi'(v) \geq 3$ .

This completes the proof. ■

**Lemma 2.3.3** *Let  $t$  and  $m$  be integers such such  $t \geq 108900$  and  $3 \leq m \leq 6$ . Let  $G$  be a graph on  $n = t + m$  vertices with*

$$e(G) > \frac{1}{2}(t + 3)(n - 2) + 1.$$

*Then  $G$  contains a  $K_{3,t}$ -minor.*

*Proof.* We consider separately each value of  $m$ .

If  $m = 3$ , then the edge condition guarantees three dominating vertices in  $G$ , so  $G$  contains a  $K_{3,t}$ -subgraph.

If  $m = 4$ , then  $\overline{G}$  contains less than  $n - 2$  edges, so it has at least 3 components that are trees. By Proposition 2.3.1 we may assume  $\overline{G}$  has no isolated vertices. Let  $x_1$  and  $x_2$  be vertices of degree 1 in different components in  $\overline{G}$ , and let  $y_i$  be the neighbor of  $x_i$ . In  $G$ , if we contract  $y_1$  and  $y_2$  into

a new vertex  $y$ , then  $x_1$ ,  $x_2$ , and  $y$  have  $t$  common neighbors, so  $G$  has a  $K_{3,t}$ -minor.

If  $m = 5$ , we are done by Lemma 2.3.2.

If  $m = 6$ , we have that  $e(\overline{G}) < 2n - 4$ . If any vertex had degree less than  $(t+3)/2$  in  $G$ , we could delete it and apply the argument for  $m = 5$ . Therefore we can assume  $d_G(v) \geq (t+3)/2$  for all  $v \in V(G)$ , so  $\Delta(\overline{G}) \leq (t+7)/2 = (n+1)/2$ . By a result of Erdős, Rényi, and Sós [25], the maximum number of edges in a graph of diameter 2 and maximum degree at most  $11n/20$  is  $2.9n - 11$ . Let  $S \subset V(G)$  have cardinality at most 4. Then

$$e(\overline{G} - S) \leq 2n - 4 < 2.9(n - |S|) - 11$$

and

$$\Delta(\overline{G} - S) \leq \frac{n+1}{2} \leq \frac{11(n - |S|)}{20},$$

so  $G - S$  cannot have diameter 2. Therefore we can find three disjoint pairs  $(v_1^i, v_2^i)$  for  $i \in \{1, 2, 3\}$  with  $\text{dist}_{\overline{G}}(v_1^i, v_2^i) \geq 3$  for each  $i$ . Contracting each edge  $v_1^i v_2^i$  in  $G$  gives  $K_{3,t}$  as a minor of  $G$ . ■

A  $\Delta$ -system is a hypergraph with edge set  $\{A_1, \dots, A_m\}$  and a set  $K$  such that  $A_i \cap A_j = K$  whenever  $i \neq j$ . The set  $K$  is called the *kernel* of the  $\Delta$ -system.

We will need the following Lemma on  $\Delta$ -systems, commonly known as the Sunflower Lemma.

**Lemma 2.3.4** ([24]) *Let  $\mathcal{H}$  be a hypergraph whose edges all have cardinality at most  $r$ . If  $\mathcal{H}$  has at least  $r!(k-1)^r$  edges, it contains a  $\Delta$ -system with  $k$  edges.*

**Lemma 2.3.5** *Let  $t$ ,  $s$ , and  $d$  be positive integers satisfying*

$$t \geq ((d+1)!(2s)^{d+1})^2.$$

Let  $H$  be a graph of order  $n = t + 2s + d$ . Suppose that  $e(H) < \frac{1}{2}(d + 4)n$  and that  $\Delta(H) \leq t/2 + d$ . Then there is a set  $D$  such that

- (1)  $|D| \leq d$  and  $H - D$  has  $s$  disjoint pairs of vertices at distance at least 3,  
or  
(2)  $|D| \leq d + 1$  and  $H - D$  has  $s$  isolated vertices.

*Proof.* Suppose first that there are at least  $2(d + 1)!(2s)^{d+1}$  vertices with degree at most  $d + 1$  in  $H$ . Then there is some  $i \leq d + 1$  such that  $H$  has at least  $i!(2s)^i$  vertices of degree  $i$ . Let  $\mathcal{N}_{\mathcal{H}}$  be the hypergraph with vertex set  $V(H)$  and edge set  $\{N_H(v) : d_H(v) = i\}$  (edges appearing with multiplicity). By Lemma 2.3.4,  $\mathcal{N}_{\mathcal{H}}$  contains a  $\Delta$ -system on  $2s + 1$  sets; let these sets be  $N_H(v_j)$  for  $1 \leq j \leq 2s + 1$ . Suppose that the kernel  $K$  of the  $\Delta$ -system has cardinality less than  $i$ . Since  $v_j$  can only be contained in  $N_H(v_k)$  for one value of  $k$ , there is a matching  $M$  on  $s$  edges in  $\overline{H}[v_1, \dots, v_{2s+1}]$ . Let  $D = K$ . Then the endpoints of the edges of  $M$  witness that  $H$  satisfies (1). If  $K$  has cardinality  $i$ , then letting  $D = K$ , we have that  $v_1, \dots, v_s$  are isolated in  $H - D$ .

Let  $V_0 \subseteq V(H)$  be the set of vertices of degree at most  $d + 1$ . From the last paragraph, we may assume  $|V_0| \leq 2(d + 1)!(2s)^{d+1} \leq \sqrt{n}$ . Let  $N^2(v)$  be the set of vertices at distance 1 or 2 from  $v$  in  $H$ . Then

$$S(H) := \sum_{v \in V(H)} |N_H^2(v)| \leq \sum_{v \in V(H)} \sum_{u \in N_H(v)} d_H(u) = \sum_{v \in V(H)} d_H(v)^2.$$

Since  $x^2$  is a convex function, the sum on the right is maximized when each vertex of  $V_0$  has degree 0 and all but at most one vertex of  $V(H) - V_0$  has degree  $d + 2$  or  $\Delta(H)$ . Therefore,

$$S(H) \leq (d + 2)^2 n + (((d + 4) - (d + 2))n + (d + 2)\sqrt{n})\Delta(H),$$

which is less than  $.91n^2$  (using  $d + 2 \leq \sqrt{n}/36$ ) unless  $H$  has a vertex  $x$  of degree at least  $.43n$ . In that case,

$$S(H - x) \leq (d + 2)^2 n + (2n - d_H(x) + (d + 2)\sqrt{n})\Delta(H)$$

$$\leq (d+2)^2n + (1.57n + (d+2)\sqrt{n})(t/2 + d) \leq .91n^2.$$

Let  $D = \{x\}$  if  $x$  exists and  $D = \emptyset$  otherwise (that is, if  $\Delta(H) \leq .43n$ ). Since  $S(H - D) \leq .92(n-1)(n-2)$ , there must be  $s$  vertices  $v_1^1, \dots, v_1^s$  in  $H - D$  for which  $|N^2(v_1^i)| \leq .95n$ . Then we can find  $s$  distinct vertices  $v_2^1, \dots, v_2^s$  in  $H - \{v_1^1, \dots, v_1^s\} - D$  such that  $\text{dist}_H(v_1^i, v_2^i) \geq 3$  for  $1 \leq i \leq s$ . But then  $H$  satisfies (1), so we are done. ■

**Lemma 2.3.6** *Let  $t, s,$  and  $d$  be positive integers satisfying*

$$t \geq ((d+1)!(2s)^{d+1})^2.$$

*Let  $G$  be a graph of order  $n = t + 2s + d$  which is minor-minimal with respect to the condition*

$$e(G) > \frac{1}{2}(t+3)(n-2) + 1.$$

*Then  $G$  contains a  $K_{3,t}$ -minor.*

*Proof.* Let  $G$  be as above. This implies that  $G$  has minimum degree at least  $(t+3s+1)/2$ , so  $\Delta(\overline{G}) \leq t/2 + d$ . Also,

$$e(\overline{G}) < \frac{1}{2}(d+4)n(G).$$

By Lemma 2.3.5, there is a subset  $D \subseteq V(G)$  such that  $|D| \leq d$  and  $\overline{G} - D$  has 3 disjoint pairs of vertices at distance at least 3 or  $|D| \leq d+1$  and  $\overline{G} - D$  has 3 isolated vertices. In the first case, contracting the three pairs in  $G$  gives the vertices of the smaller partite set of a  $K_{3,t}$ -minor. In the second, the three isolated vertices in  $\overline{G} - D$  are the smaller partite set of a  $K_{3,t}$ -subgraph of  $G$ . ■

*Proof of Theorem 2.1.6.* Let  $G'$  be a minor-minimal subgraph of  $G$  with respect to the edge condition. If  $n(G') \leq t+6$ , apply Lemma 2.3.3. Otherwise, if  $n \leq t+32$ , apply Lemma 2.3.6 to  $G'$  with  $s = 3$ . If  $t+33 \leq n(G') \leq 10t/9$ , apply Lemma 2.2.6. Finally, if  $n > 10t/9$ , we can apply Lemma 2.2.14 to  $G'$ .

Either way,  $G'$ , and hence  $G$ , contains a  $K_{3,t}$ -minor. □

Note that Theorem 2.1.6 did not guarantee the existence of a  $K_{3,t}^*$ -minor in  $G$ . This is because the edge bound of Theorem 2.1.6 is insufficient to force a  $K_{3,t}^*$ -minor. To see this, notice that if  $H = mK_2$  and  $t = 2m - 4$ , then  $G = \overline{H}$  has no  $K_{3,t}^*$ -minor but satisfies

$$e(G) = \frac{1}{2}(t+2)n > \frac{1}{2}(t+3)(n-2) + 1.$$

To get larger examples (that is, with  $n$  larger with respect to  $t$ ), one can take many copies of  $G$  and identify all the copies of a single non-edge  $uv \notin E(G)$ .

# Chapter 3

## Domination

### 3.1 Introduction

A *dominating set* of a graph  $G$  is a set  $D \subseteq V(G)$  such that every vertex of  $V(G) - D$  has a neighbor in  $D$ . The *domination number* of  $G$ , written  $\gamma(G)$  is the smallest size of a dominating set of  $G$ .

Domination is the subdiscipline of graph theory that studies the sizes and structures of dominating sets in graphs. The book [28] provides an excellent extensive introduction to the field, and its companion text [27] surveys many related advanced topics.

Much of the contemporary study of domination centers on generalizations of the domination problem. One way to generalize the problem is to redesignate *which vertices* are to be dominated. For example, a *total dominating set* of  $G$  is a subset  $D \subseteq V(G)$  such that every vertex in  $V(G)$  has a neighbor in  $D$ . Another way to change the problem is to redetermine *how many neighbors* each vertex of  $G$  should have in the dominating set. In this direction we have *k-dominating sets*, where every vertex not in the set has at least  $k$  neighbors in the set (a dominating set is, then, a 1-dominating set).

One recent trend in domination is the study of dominating functions. Dominating functions stem from the idea of considering  $D \subseteq V(G)$  not as a

set, but rather as its characteristic function  $\chi_D$  ( $\chi_D(v)$  is 1 if  $v \in D$  and 0 otherwise). Therefore a dominating set  $D$  in  $G$  is one for which

$$\sum_{u \in N_G[v]} \chi_D(u) \geq 1 \tag{3.1}$$

for every vertex  $v \in V(G)$ . If we define the *weight* of a function  $f : V(G) \rightarrow \mathbb{R}$  to be  $\sum_{v \in V(G)} f(v)$ , then  $\gamma(G)$  is the minimum weight of a dominating function of  $G$ .

Variations of dominating functions arise by altering the defining condition (that corresponding to Equation 3.1). For example, a *Roman dominating function* (hereafter RDF) of  $G$  is a map  $f : V(G) \rightarrow \{0, 1, 2\}$  such that every vertex in  $f^{-1}(0)$  has a neighbor in  $f^{-1}(2)$ . The *Roman domination number* of  $G$ , written  $\gamma_R(G)$ , is the minimum weight of an RDF of  $G$ . Note that the vertices of positive weight in an RDF of  $G$  are a dominating set, so  $\gamma_R(G) \geq \gamma(G)$ . On the other side, the RDF giving weight 2 to the vertices of a minimum dominating set and 0 to all other vertices has weight  $2\gamma(G)$ , so  $\gamma_R(G) \leq 2\gamma(G)$ .

One way to study the extremal behavior of a parameter  $P(G)$  is to consider the relationship between  $P(G)$  and  $P(\overline{G})$ , where  $\overline{G}$  is the complement of  $G$ . *Nordhaus-Gaddum* results for  $P$  refer to upper and lower bounds on  $P(G) + P(\overline{G})$  or  $P(G)P(\overline{G})$ , usually in terms of the order of  $G$ . Relating  $G$  and  $\overline{G}$  like this gives results of this type a Ramsey flavor. Such bounds have been derived for many graph parameters, including domination variants. For example, Jaeger and Payan derived the following result for  $\gamma(G)$ .

**Theorem 3.1.1** ([31]) *For any graph  $G$  on  $n$  vertices,*

$$\gamma(G) + \gamma(\overline{G}) \leq n + 1$$

$$\gamma(G)\gamma(\overline{G}) \leq n.$$

In this chapter, we will study the extremal behavior of certain domination parameters. The first parameter we consider, in Section 3.2, is  $\gamma(G)$  itself.

We reprove a Nordhaus-Gaddum bound for  $\gamma(G)$  due to Jaeger and Payan. Then, in Section 3.3, we extend that result to  $\gamma_k(G)$  for  $k > 1$ . Section 3.4 sees several applications of the methods of Sections 3.2 and 3.3 to parameters which behave like  $\gamma(G)$  or  $\gamma_k(G)$ . Finally, in Section 3.5, we take another approach, studying when the Roman domination number  $\gamma_R(G)$  is close to  $\gamma(G)$  or to  $2\gamma(G)$  (it is trivially bounded between the two).

## 3.2 Nordhaus-Gaddum Bounds for Domination

In this section, we reprove Theorem 3.1.1. The original proof of Jaeger and Payan was both short and elegant. However, our proof has the advantage of being easily adapted to derive similar results not just for the domination number of  $G$ , but also for similar parameters. Indeed, this method was developed in [16] for use with Roman domination. This method will be extended to  $k$ -domination in the next Section, and in Section 3.4 we demonstrate how to apply these techniques to domination-like parameters.

The idea of the proof is to show that  $\gamma(G)\gamma(\overline{G})$  is maximized when  $\text{diam}(G)$  or  $\text{diam}(\overline{G})$  is at least 3. From there, a short ad hoc argument suffices to finish the proof.

One key lemma we will need is due to Arnautov [3] and Payan [43], which was reproved by an application of the probabilistic method in [2].

**Lemma 3.2.1 ([2])** *Let  $G$  be a graph of order  $n$  with  $\delta(G) = k$ . Then*

$$\gamma(G) \leq \frac{\ln(k+1) + 1}{k+1} n.$$

We are now ready to show that  $\gamma(G)\gamma(\overline{G})$  is small when  $\text{diam}(G)$  and  $\text{diam}(\overline{G})$  are at most 2.

**Theorem 3.2.2** *Let  $G$  be a graph of order  $n$  such that both  $\text{diam}(G)$  and  $\text{diam}(\overline{G})$  are at most 2. Then*

$$\gamma(G)\gamma(\overline{G}) \leq 4(n \ln(n))^{2/3}.$$

*Proof.* Let  $G$  be such a graph, and let  $v$  be a vertex of minimum degree in  $G$ . Let  $R = V(G) - N_G[v]$ .

Define sets  $A_1, \dots, A_q$  and  $A^*$  as follows: while  $B_j = N_G(v) - \bigcup_{i < j} A_i$  dominates  $R$ , let  $A_j$  be a minimal subset of  $B_j$  that dominates  $R$  and increment  $j$ . If  $B_j$  does not dominate  $R$ , then let  $A^* = B_j$  and let  $q = j - 1$ .

Since  $A_i$  is a minimal dominating set for  $R$ , there is an  $r_i \in R$  which is adjacent to only one vertex, say  $a_i$ , in  $A_i$ . Also, since  $A^*$  does not dominate  $R$ , there is a vertex  $r^* \in R$  with no neighbors in  $A^*$ .

Now,  $v$  together with the smallest  $A_i$  form a dominating set of  $G$ , so

$$\gamma(G) \leq 1 + \min_i (|A_i|) \leq 1 + d_G(v)/q.$$

Also, the set  $\{r_i, a_i : 1 \leq i \leq q\} \cup \{r^*, v\}$  is a dominating set of  $\overline{G}$ , so  $\gamma(\overline{G}) \leq 2q + 2$ . Therefore

$$\gamma(G)\gamma(\overline{G}) \leq (1 + d_G(v)/q)(2q + 2) \leq 4d_G(v) + 4,$$

which is less than  $4(n \ln(n))^{2/3}$  if  $d_G(v) \leq (n \ln(n))^{2/3} - 1$ .

We may therefore assume that  $\delta(G)$ , and by symmetry  $\delta(\overline{G})$ , is at least  $(n \ln(n))^{2/3} - 1$ . Apply Lemma 3.2.1 to see that  $\gamma(G)$  and  $\gamma(\overline{G})$  are at most

$$\frac{\ln((n \ln(n))^{2/3}) + 1}{(n \ln(n))^{2/3}} n \leq 2(n \ln(n))^{1/3},$$

so

$$\gamma(G)\gamma(\overline{G}) \leq \left(2(n \ln(n))^{1/3}\right)^2 = 4(n \ln(n))^{2/3},$$

and we are done. ■

**Theorem 3.2.3** *Let  $G$  be a graph of order  $n \geq 4538$ . Then*

$$\gamma(G)\gamma(\overline{G}) \leq n.$$

*Furthermore, equality holds only if  $\{\gamma(G), \gamma(\overline{G})\}$  is  $\{1, n\}$  or  $\{2, n/2\}$ .*

*Proof.* If  $G$  and  $\overline{G}$  have diameter at most 2, then

$$\gamma(G)\gamma(\overline{G}) \leq 4(n \ln(n))^{2/3} < n.$$

We may therefore assume, by symmetry, that  $\text{diam}(G) \geq 3$ .

If  $G$  has an isolated vertex, then  $\gamma(\overline{G}) = 1$ , so

$$\gamma(G)\gamma(\overline{G}) \leq n,$$

with equality only if  $\gamma(G) = n$ .

Based on the last paragraph, we may assume that  $G$  has minimum degree at least 1. It is well-known that in this case  $\gamma(G) \leq n/2$ , since if  $A$  is a minimum dominating set in  $G$ , then  $V(G) - A$  is a dominating set of  $G$  as well. If  $u$  and  $v$  are vertices at distance at least 3 in  $G$ , then  $\{u, v\}$  is a dominating set of  $\overline{G}$ , so  $\gamma(\overline{G}) \leq 2$ . Therefore

$$\gamma(G)\gamma(\overline{G}) \leq n,$$

with equality only if  $\gamma(G) = n/2$ . ■

Note two differences between Theorems 3.2.3 and 3.1.1. First, Theorem 3.1.1 applies for all  $n$ , whereas Theorem 3.2.3 only applies when  $n$  is sufficiently large. Second, Theorem 3.1.1 does not characterize equality, but Theorem 3.2.3 does.

### 3.3 Nordhaus-Gaddum Bounds for $k$ - Domination

For a graph  $G$ , a set  $D \subseteq V(G)$  is a  $k$ -dominating set of  $G$  if every vertex in  $V(G) - D$  has at least  $k$  neighbors in  $D$ . The  $k$ -domination number of  $G$ ,

written  $\gamma_k(G)$ , is the minimum size of a  $k$ -dominating set of  $G$ .

For any graph  $G$  on  $n \geq k$  vertices, we have trivially that  $k \leq \gamma_k(G) \leq n$ . Equality in the upper bound is achieved whenever the maximum degree of  $G$  is less than  $k$ . If we rule out small degree vertices, this bound can be improved.

**Theorem 3.3.1** ([22, 14]) *If  $G$  is a graph of order  $n$  with minimum degree at least  $k$ , then  $\gamma_k(G) \leq kn/(k+1)$ .*

The goal of this section is to establish Nordhaus-Gaddum bounds for  $k$ -domination, thereby generalizing Jaeger and Payan's Theorem 3.1.1 to  $k > 1$ .

**Main Result.** *For any integer  $k \geq 2$  and any graph  $G$  of order  $n \geq n_0(k)$ ,*

$$8k/3 \leq \gamma_k(G) + \gamma_k(\overline{G}) \leq n + 2k - 1$$

$$16k^2/9 \leq \gamma_k(G)\gamma_k(\overline{G}) \leq (2k-1)(n-k+2).$$

*Furthermore, these bounds are sharp.*

The Main Result will be established in several pieces, namely, Theorems 3.3.2, 3.3.13, and 3.3.17.

The function  $n_0(k)$  appearing in the Main Result is small for three of the inequalities. Indeed, the lower bounds hold already for  $n \geq 25k/9$ , and the upper bound for sums is valid for all  $n$ . Our argument for the upper bound on products, however, requires  $n$  to be at least  $\exp(\Omega(k \log k))$ . For smaller values of  $n$ , we can prove a weaker product upper bound of the same order of magnitude (that is,  $O(kn)$ ).

When the ratio  $\Delta(G)/\delta(G)$  is small enough (for example, for regular graphs) and  $n$  is large, we can improve the product upper bound to

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (k+1)n,$$

and if in addition  $\delta(G) \geq k$ , then

$$\gamma_k(G)\gamma_k(\overline{G}) \leq kn.$$

This constitutes Theorem 3.3.18.

In the case  $k = 2$ , we can use some of our results to prove a conjecture of Alon, Balogh, Bollobás, and Szabó concerning game domination.

The proofs of these results employ a variety of methods. Most involve combinatorial reasoning aided by the probabilistic method. The upper bound on products, though, introduces a novel approach based on  $\Delta$ -systems.

The organization of this section is as follows. In subsection 3.3.1, we derive the lower Nordhaus-Gaddum bounds for  $\gamma_k$ . In subsections 3.3.2 and 3.3.3, we prove a handful of lemmas and propositions that will be needed for the upper bounds. In subsection 3.3.4, we develop  $\Delta$ -systems as a tool for subsection 3.3.5, where we derive both the upper Nordhaus-Gaddum bounds for  $\gamma_k$  mentioned in the Main Result and a weaker product bound valid for all values of  $n$ . Finally, subsection 3.3.6 improves the bounds of the subsection prior when the ratio  $\Delta(G)/\delta(G)$  is small.

### 3.3.1 Lower Bounds

In this section we present the following theorem giving Nordhaus-Gaddum-type lower bounds for  $\gamma_k$ .

**Theorem 3.3.2** *Let  $G$  be a graph of order  $n \geq 25k/9$ . Then*

$$\gamma_k(G) + \gamma_k(\overline{G}) \geq 8k/3$$

and

$$\gamma_k(G)\gamma_k(\overline{G}) \geq 16k^2/9.$$

*Proof.* Let  $A$  be a minimum  $k$ -dominating set of  $G$ , let  $B$  be a minimum  $k$ -dominating set of  $\overline{G}$ , and let  $C = A \cap B$ . Let  $a = |A|$ ,  $b = |B|$ , and  $c = |C|$ .

Suppose first that  $A \cup B = V(G)$ . Then, because  $\gamma_k$  is at least  $k$  for  $G$  and  $\overline{G}$ , we have

$$\gamma_k(G)\gamma_k(\overline{G}) \geq k(n-k) \geq 16k^2/9,$$

which forces  $\gamma_k(G) + \gamma_k(\overline{G}) \geq 8k/3$ , so the desired bounds hold. We therefore assume  $A \cup B \neq V(G)$ . Then, since every vertex not in  $A \cup B$  sends  $2k$  edges into  $A \cup B$  between  $G$  and  $\overline{G}$ , we see that  $|A \cup B| = a + b - c \geq 2k$ .

Now suppose that  $A = B = C$ . Then, since each vertex outside  $C$  sends  $2k$  edges into  $C$  between  $G$  and  $\overline{G}$ , we have  $c \geq 2k$ , so

$$\gamma_k(G) + \gamma_k(\overline{G}) = 2c \geq 4k$$

and

$$\gamma_k(G)\gamma_k(\overline{G}) = c^2 \geq 4k^2.$$

We may therefore assume  $A \neq B$ , so that in particular  $a + b - 2c \neq 0$ .

Now, in  $G$ , each vertex of  $B - A$  has at least  $k$  neighbors in  $A$ , and in  $\overline{G}$  each vertex of  $A - B$  has at least  $k$  neighbors in  $B$ . Therefore, by counting edges, we have

$$(a-c)k + (b-c)k \leq |A-B||B| + |B-A||A| - |A-B||B-A|,$$

which is equivalent to

$$(a-c)k + (b-c)k \leq ab - c^2. \tag{3.2}$$

But then

$$ab \geq (a+b-2c)k + c^2 \geq (2k-c)k + c^2. \tag{3.3}$$

We also know that in  $G$  each vertex of  $A - C$  has at least  $k - c$  neighbors in  $B - C$ , and in  $\overline{G}$  each vertex of  $B - C$  has at least  $k - c$  neighbors in  $A - C$ . By counting possible edges between  $A - C$  and  $B - C$ , we have

$$(k-c)(a+b-2c) \leq (a-c)(b-c) \leq \frac{1}{4}(a+b-2c)^2.$$

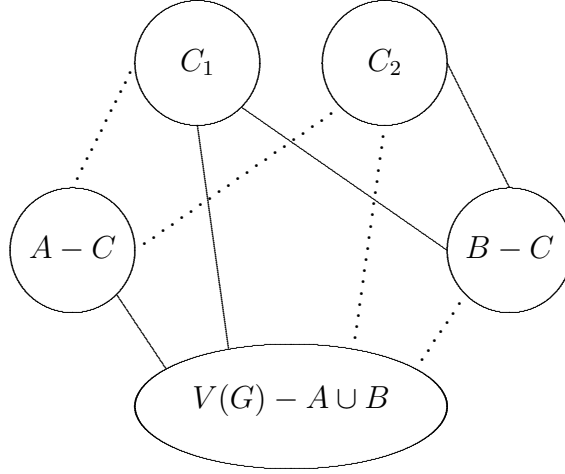


Figure 3.1: Theorem 3.3.2 is sharp.

Dividing through by  $a + b - 2c$  shows  $a + b - 2c \geq 4k - 4c$ . Combining this with Equation 3.2 yields

$$ab \geq k(4k - 4c) + c^2. \quad (3.4)$$

The maximum of the two quadratics in  $c$  from Equations 3.3 and 3.4 is minimized at  $c = 2k/3$ , giving us that  $ab = \gamma_k(G)\gamma_k(\overline{G}) \geq 16k^2/9$ . This immediately implies that  $\gamma_k(G) + \gamma_k(\overline{G}) \geq 8k/3$ , so we are done. ■

The bounds in Theorem 3.3.2 are sharp when  $k$  is divisible by 3, as shown by the example of Figure 3.1. Let  $A$  and  $B$  be sets of  $4k/3$  vertices whose intersection  $C$  is the disjoint union of two sets  $C_1$  and  $C_2$ , each of size  $k/3$ . Lines between sets represent edges in  $G$ , whereas dotted lines are edges of  $\overline{G}$ . Let  $A - C$  and  $B - C$  be the partite sets of an induced  $k/3$ -regular bipartite graph in  $G$  (and, by symmetry, in  $\overline{G}$ ). All other edges are arbitrary. It is easy to see  $A$  and  $B$  are  $k$ -dominating sets for  $G$  and  $\overline{G}$ , respectively.

### 3.3.2 Some Lemmas

We now set our sights on proving upper bounds. Before we can, though, we must gather several lemmas. The first two are applications of the probabilistic method. We use  $\mathbb{P}$  to denote probability and  $\mathbb{E}$  for expectation.

**Theorem 3.3.3** *Let  $k \geq 2$  be an integer and  $G$  be a graph of order  $n$ . Suppose that at most  $m$  vertices of  $G$  have degree less than  $d \geq 7k \log k$ . Then*

$$\gamma_k(G) \leq \left( \frac{k \log d}{d} + \frac{k(2e \log d)^{k-1}}{d^k} \right) n + m.$$

*Proof.* Let  $M$  be the set of vertices of degree less than  $d$  in  $G$ . Choose a set  $X \subseteq V(G)$  by putting each vertex of  $G$  into  $X$  independently with probability  $p = k \log d/d$ , so  $\mathbb{E}(|X|) = pn$ . Note that since  $d \geq 7k \log k$ , we have  $p < 1/2$ .

Let  $Y$  be the set of vertices of  $V(G) - M$  with fewer than  $k$  neighbors in  $X$ . For each vertex  $v \in V(G) - M$ , let  $N_v$  be a set of  $d$  neighbors of  $v$ . Then

$$\begin{aligned} \mathbb{P}(v \in Y) &\leq \mathbb{P}(|N_v \cap X| < k) = \sum_{i=0}^{k-1} \binom{d}{i} p^i (1-p)^{d-i} \\ &\leq (1-p)^d \sum_{i=0}^{k-1} \frac{(dp)^i}{i!} \left( \frac{1}{1-p} \right)^i < \frac{2^{k-1}}{d^k} \sum_{i=0}^{k-1} \frac{(k \log d)^i}{i!} \end{aligned}$$

using that  $(1-p)^d \leq e^{-pd} = 1/d^k$  and  $1/(1-p) < 2$ . The maximum term of the sum is that corresponding to  $i = k-1$ , so the sum is at most

$$k \frac{(k \log d)^{k-1}}{(k-1)!} \leq k(2e \log d)^{k-1}.$$

Therefore

$$\mathbb{E}(Y) = n \cdot \mathbb{P}(v \in Y) \leq n \cdot \frac{k(2e \log d)^{k-1}}{d^k}.$$

Since  $X \cup M \cup Y$  is a  $k$ -dominating set with expected size  $pn + m + \mathbb{E}(Y)$ , there is some  $k$ -dominating set with at most the desired size.  $\blacksquare$

It should be noted that when  $d$  is much larger than  $k$ , then setting  $p = (\log d + (k - 1) \log \log d)/d$  gives a better bound. However, for the small values of  $d$  with which we are concerned, the ease of presentation justifies any difference in the bounds.

**Lemma 3.3.4** *Let  $k \geq 2$  be an integer and let  $G$  be a graph of order  $n$ . Suppose there is a set of  $q$  vertices which  $k$ -dominates all vertices of  $G$  with degree less than  $40k$ . Then  $\gamma_k(G) < 21n/100 + q$ .*

*Proof.* Repeat the proof of Lemma 3.3.3 with  $p = 1/5$ , so  $\mathbb{E}(X) = n/5$ . Let  $Y$  be the set of vertices with degree at least  $d = 40k$  which are not  $k$ -dominated by  $X$ . Then

$$\mathbb{P}(v \in Y) \leq k \left(\frac{4}{5}\right)^d \frac{(ed)^{k-1}}{(4(k-1))^{k-1}} \leq k \left(\frac{4}{5}\right)^d (10e)^{k-1} < \frac{k}{100}.$$

This shows that  $\mathbb{E}(|X| + |Y|) < 21n/100$ , and adding  $q$  vertices to  $k$ -dominate the small-degree vertices shows there is a  $k$ -dominating set with at most the desired size. ■

We can, of course, apply the lemma above with  $q$  equal to the number of vertices of degree less than  $40k$ . It will be advantageous during the proof of Lemmas 3.3.5 and 3.3.16, though, to exploit the additional generality we have afforded ourselves.

**Lemma 3.3.5** *Let  $G$  be a graph of order  $n \geq 6400$  with  $\gamma_2(\overline{G}) \leq 7$  and  $\gamma_2(G)\gamma_2(\overline{G}) > 2n$ . Then  $\gamma_2(\overline{G}) \leq 4$ .*

*Proof.* Let  $M = \{v : d_G(v) < 80\}$ . The hypotheses imply that  $\gamma_2(G) > 2n/7$ , so by Lemma 3.3.4 with  $q = |M|$ , we have  $|M| > n/20$ . Among pairs of vertices of  $M$ , let  $v_1$  and  $v_2$  minimize  $|N_G(v_1) \cap N_G(v_2)|$ .

If  $U = N_G(v_1) \cup N_G(v_2)$  2-dominates all but 160 vertices of  $M$ , then by Lemma 3.3.4

$$\gamma_2(G) \leq 21n/100 + |U| + 160 \leq 2n/7,$$

a contradiction. Hence at least 161 vertices of  $M$  have at most one neighbor in  $U$ , and so two of them, say  $v_3$  and  $v_4$ , are nonadjacent to all of  $U - x$  for some  $x \in U$ . Since  $|N_G(v_1) \cap N_G(v_3)| \leq 1$ , we see by minimality that  $v_1$  and  $v_2$  have at most one common neighbor.

Now, in  $\overline{G}$ ,  $v_1$  and  $v_3$  2-dominate  $V(G) - N_G(v_1) - N_G(v_3)$  and  $v_i$  and  $v_4$  2-dominate  $N_G(v_{4-i}) - x$  for  $i = 1, 3$ . This shows that  $\{v_1, v_3, v_4, x\}$  is a 2-dominating set of size 4 in  $\overline{G}$ , which completes the proof. ■

Our fourth lemma is an easy corollary of the following theorem of Erdős, Faudree, Gyárfás, and Schelp.

**Theorem 3.3.6** ([23]) *For any graph  $G$  on  $n$  vertices, either  $\gamma(G)$  or  $\gamma(\overline{G})$  is at most  $\lceil \log n \rceil$ .*

**Lemma 3.3.7** *For any graph  $G$  on  $n$  vertices and any integer  $k \geq 1$ , either  $\gamma_k(G)$  or  $\gamma_k(\overline{G})$  is at most  $(2k - 1)\lceil \log n \rceil$ .*

*Proof.* Inductively define  $X_i$  to be a dominating set of  $G - \bigcup_{j=1}^{i-1} X_j$  or  $\overline{G} - \bigcup_{j=1}^{i-1} X_j$  of size at most  $\lceil \log n \rceil$ , which exists by Theorem 3.3.6. Let  $X = \bigcup_{j=1}^{2k-1} X_j$ .

By the Pigeonhole Principle, at least  $k$   $X_i$ 's dominate  $G - X$  or at least  $k$   $X_i$ 's dominate  $\overline{G} - X$ , so either  $\gamma_k(G)$  or  $\gamma_k(\overline{G})$  is at most  $|X| \leq (2k - 1)\lceil \log n \rceil$ . ■

The final lemma of this section comes essentially from [14], where it was proven for  $m = 0$ . The proof is practically the same; we omit it.

**Lemma 3.3.8** ([14]) *Let  $w$  and  $k$  be positive integers. If  $G$  is a graph of order  $n$  with at most  $m$  vertices of degree less than  $(w + 1)k/w - 1$ , then  $\gamma_k(G) \leq wn/(w + 1) + m$ .*

### 3.3.3 Some Computations

In this section we collect two propositions whose proofs involve simple computation. Doing this work here will facilitate the proof of Lemma 3.3.16.

**Proposition 3.3.9** *Let  $r$  and  $k$  be positive integers satisfying  $r \geq k$ . Let  $w = \lfloor r/(r - k + 1) \rfloor$ . Then  $\lfloor (w + 1)k/w - 1 \rfloor < r$ .*

*Proof.* By definition  $\lfloor (w + 1)k/w - 1 \rfloor = k - 1 + \lfloor k/w \rfloor$ , so we are done as long as  $\lfloor k/w \rfloor < r - k + 1$ . This is true if  $k/w < r - k + 1$ , which is equivalent to

$$\frac{k}{r - k + 1} < \left\lfloor \frac{r}{r - k + 1} \right\rfloor = 1 + \left\lfloor \frac{k + 1}{r - k + 1} \right\rfloor.$$

This holds because  $\lfloor x \rfloor > x - 1$  for all real  $x$ . ■

**Proposition 3.3.10** *Let  $r$  and  $k$  be positive integers satisfying*

$$2k - 1 \geq r \geq k \geq 2.$$

*Let  $w = \lfloor r/(r - k + 1) \rfloor$  and  $m = (k + r)w/(w + 1)$ . Then  $m \leq 2k - 4/3$ .*

*Proof.* First we consider the cases when  $w \leq 3$ . If  $w = 1$ , then  $r = 2k - 1$ , so  $m = (3k - 1)/2 < 2k - 4/3$ . If  $w = 2$ , then  $r \leq 2k - 2$  so  $m \leq 2(3k - 2)/3 = 2k - 4/3$ . If  $w = 3$ , then  $r \leq 3(k - 1)/2$ , so  $m \leq 15k/8 - 9/8 < 2k - 4/3$ . In all three cases, the desired bound holds.

Now consider  $w \geq 4$ , so that  $r \leq 4(k - 1)/3$ . Then we have

$$\begin{aligned} m &= (k + r) \left( 1 - \frac{1}{w + 1} \right) = (k + r) \left( 1 - \frac{1}{\lfloor (2r - k + 1)/(r - k + 1) \rfloor} \right) \\ &\leq (k + r) \frac{r}{2r - k + 1}, \end{aligned}$$

where the inequality comes from ignoring the floor signs in the denominator. The derivative (with respect to  $r$ ) of the right hand side above is negative

when  $r \leq 4(k-1)/3$ , so  $m$  is maximized when  $r$  is as small as possible, namely  $r = k$ . At  $r = k$  we have  $w = k$  and  $m = 2k^2/(k+1) \leq 2k - 4/3$ , which proves the proposition. ■

### 3.3.4 $\Delta$ -systems

A  $\Delta$ -system is a set system  $\mathcal{A} = \{A_1, \dots, A_\ell\}$  such that there is a set  $K$  with  $A_i \cap A_j = K$  whenever  $i \neq j$ .  $K$  is called the *kernel* of  $\mathcal{A}$ . An  $(r, \ell)$ - $\Delta$ -system is one which additionally satisfies  $|\mathcal{A}| = \ell$  and  $|A| = r$  for all  $A \in \mathcal{A}$ . For a survey of results on  $\Delta$ -systems, see [34].

The most famous question on  $(r, \ell)$ - $\Delta$ -systems is how large an  $r$ -uniform set system must be in order to contain one. Erdős and Rado gave an upper bound, which we stated as Theorem 2.3.4.

Recall from Section 1.2 the definition of neighborhood hypergraph of  $G$ ,  $\mathcal{N}_G$ , whose vertex set is  $V(G)$  and whose hyperedges are the sets  $N_G(v)$  appearing with multiplicity. We shall apply Theorem 2.3.4 to find  $(r, \ell)$ - $\Delta$ -systems in  $\mathcal{N}_G$ . To this end, we make the following observation.

**Proposition 3.3.11** *Let  $r$ ,  $k$ , and  $q$  be positive integers and let  $G$  be a graph with at least  $2(kq)^q$  vertices of degree at most  $q$ . Then  $\mathcal{N}_G$  contains an  $(r, k+1)$ - $\Delta$ -system for some  $r \leq q$ .*

*Proof.* By our assumptions, the hypergraph  $\mathcal{N}_G^{\leq q}$  with edge set  $\{N_G(v) : d_G(v) \leq q\}$  (edges appearing with multiplicities) satisfies

$$|\mathcal{N}_G^{\leq q}| \geq 2(kq)^q > \sum_{r=0}^q k^r r!,$$

so by the pigeonhole principle and Theorem 2.3.4,  $\mathcal{N}_G$  must contain an  $(r, k+1)$ - $\Delta$ -system for some  $r \leq q$ . ■

The presence of certain  $\Delta$ -systems in  $\mathcal{N}_G$  forces  $\gamma_k(\overline{G})$  to be small.

**Proposition 3.3.12** *Suppose  $G$  is a graph and  $\mathcal{N}_G$  contains an  $(r, k + 1)$ - $\Delta$ -system with kernel  $K$ . If  $|K| = r$ , then  $\gamma_k(\overline{G}) \leq |K| + k$ . If  $|K| < r$ , then  $\gamma_k(G) \leq |K| + k + 1$ .*

*Proof.* Let  $\mathcal{A} = \{N_G(v_1), \dots, N_G(v_{k+1})\}$  be an  $(r, k + 1)$ - $\Delta$ -system in  $\mathcal{N}_G$ , and let  $K$  be the kernel of  $\mathcal{A}$ . First observe that  $\{v_1, \dots, v_{k+1}\}$   $k$ -dominates  $V(G) - K$  in  $\overline{G}$ . In light of this fact, the set  $K \cup \{v_1, \dots, v_{k+1}\}$  is a  $k$ -dominating set of  $\overline{G}$  with size  $|K| + k + 1$ . If  $|K| = r$ , then  $\{v_1, \dots, v_k\}$   $k$ -dominates  $V(G) - K$ , so  $K \cup \{v_1, \dots, v_k\}$  is a  $k$ -dominating set of  $\overline{G}$  with size  $|K| + k$ . ■

### 3.3.5 Upper Bounds

In this section we prove upper bounds on  $\gamma_k(G) + \gamma_k(\overline{G})$  and  $\gamma_k(G)\gamma_k(\overline{G})$ . We begin by proving two theorems that apply for all values of  $n(G)$ . After that, we prove two more lemmas before establishing Theorem 3.3.17, the culmination of Sections 3.3.2 through 3.3.5.

**Theorem 3.3.13** *For any graph  $G$  on  $n$  vertices,  $\gamma_k(G) + \gamma_k(\overline{G}) \leq n + 2k - 1$ .*

*Proof.* Let  $X \subseteq V(G)$  have size  $2k - 1$ . By the pigeonhole principle,  $N_G^k(X)$  and  $N_{\overline{G}}^k(X)$  are disjoint and have union  $V(G) - X$ . Therefore,  $N_G^k[X]$   $k$ -dominates  $G$  and  $N_{\overline{G}}^k[X]$   $k$ -dominates  $\overline{G}$ , so

$$\gamma_k(G) + \gamma_k(\overline{G}) \leq |N_G^k[X]| + |N_{\overline{G}}^k[X]| = n + |X| = n + 2k - 1.$$

■

Theorem 3.3.13 is sharp, at least for small  $n$ . For example, when  $k$  is odd, let  $G$  be a  $(k - 1)$ -regular graph on  $2k - 1$  vertices. Then  $\gamma_k(G) + \gamma_k(\overline{G}) = 2n = n + 2k - 1$ .

**Theorem 3.3.14** *Let  $k \geq 2$  be an integer and  $G$  be a graph of order  $n$ . Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq 8kn.$$

*Proof.* Note first that if  $\gamma_k(G) \leq 8k$  we are done, so assume otherwise. Partition  $V(G)$  into  $m = \lceil 3n/2\gamma_k(G) \rceil$  sets  $V_1, \dots, V_m$  of size at most  $2\gamma_k(G)/3$ . Note that  $m \leq 2n/\gamma_k(G)$ .

No  $V_i$   $k$ -dominates at least  $n - 2k + 3$  vertices in  $G$ , since otherwise

$$\gamma_k(G) \leq |V_i| + 2k - 3 < 3\gamma_k(G)/4 + \gamma_k(G)/4 = \gamma_k(G).$$

Therefore, for each  $i \in \{1, \dots, m\}$ , there is a set  $D_i$  of cardinality at least  $2k - 2$  such that  $|E_G(V_i, \{v\})| \leq k - 1$  for every  $v \in D_i$ . Let  $U_i$  be the subset of  $V_i$  consisting of vertices not  $k$ -dominated in  $\overline{G}$  by  $D_i$ . Then  $|U_i| \leq |E_G(V_i, D_i)|/(k - 1) \leq |D_i|$ .

Now  $D_i \cup U_i$   $k$ -dominates  $V_i$  in  $\overline{G}$ , so

$$\gamma_k(\overline{G}) \leq \left| \bigcup_{i=1}^m (D_i \cup U_i) \right| \leq (4k - 4)m \leq 4k \cdot \frac{2n}{\gamma_k(G)}.$$

Therefore  $\gamma_k(G)\gamma_k(\overline{G}) \leq 8kn$ . ■

It is likely that the constant 8 in Theorem 3.3.14 can be improved, although it is unclear how much. The complete bipartite graph  $K_{k-1, n-k+1}$  satisfies  $\gamma_k(G) = n - k + 1$  and  $\gamma_k(\overline{G}) = 2k - 1$  if  $n \geq 2k - 1$ . This tells us the constant 8 should be at least about 2. As it turns out,  $K_{k-1, n-k+1}$  is nearly optimal, but not quite as good as the following construction.

**Construction.** Let  $mG$  denote the disjoint union of  $m$  copies of  $G$ . Define the graph  $G_{n,k} = K_{k-2} \vee \frac{n-k+2}{2} K_2$ .

We claim  $\gamma_k(G_{n,k})\gamma_k(\overline{G_{n,k}}) = (2k - 1)(n - k + 2)$  if  $k$  is odd. Since  $G_{n,k}$  has only  $k - 2$  vertices of degree at least  $k$ ,  $\gamma_k(G_{n,k}) \geq n - k + 2$ , and equality clearly holds. In  $\overline{G}$ , the  $k - 2$  vertices of the copy of  $K_{k-2}$  in  $G$  must be in a

$k$ -dominating set since they have degree less than  $k$ . We need to include at least  $k + 1$  vertices from the copy of  $\frac{n-k+2}{2}K_2$ , since any  $k$  of them fail to  $k$ -dominate at least one vertex (that one vertex being one appearing in a copy of  $K_2$  in  $G$  with one of the  $k$  chosen vertices). Therefore  $\gamma_k(\overline{G}) \geq (2k - 1)$ , and it is easy to see equality holds here as well. This establishes the claim.  $\square$

We separate the bulk of the argument for Theorem 3.3.17 into the next two lemmas. Our first is a generalization of Theorem 3.2.2 to  $k > 1$ .

**Lemma 3.3.15** *Let  $G$  be a graph of order  $n \geq 10^6 k^2$ , and suppose that any  $(2k - 1)$ -subset  $X$  of  $V(G)$  satisfies that  $N_G^k(X)$   $k$ -dominates all but at most  $2k - 2$  vertices of  $V(G) - N_G^k[X]$ . Then  $\gamma_k(G)\gamma_k(\overline{G}) \leq kn$ .*

*Proof.* Let  $X$  be the set of  $2k - 1$  vertices of  $G$  with smallest degree sum. Let  $d = |N_G^k(X)|$  and  $R = V(G) - N_G^k[X] = N_G^k(X)$ .

Fix a partition  $\{A_1, \dots, A_q, A^*\}$  of  $N_G^k(X)$  such that  $A_i$  is a minimal set that  $k$ -dominates all but at most  $2k - 2$  vertices of  $R$  for  $1 \leq i \leq q$  and at least  $2k - 1$  vertices of  $R$  are not  $k$ -dominated by  $A^*$ . Note that since  $N_G^k(X)$   $k$ -dominates all but at most  $2k - 2$  vertices of  $R$  by hypothesis,  $q \geq 1$ .

Now, the union of  $X$ , the smallest  $A_i$ , and the vertices of  $R$  not  $k$ -dominated by that smallest  $A_i$  are a  $k$ -dominating set of  $G$ , so

$$\gamma_k(G) \leq |X| + \min_{1 \leq i \leq q} |A_i| + (2k - 2) \leq 4k - 3 + \frac{d}{q}.$$

In  $\overline{G}$ ,  $X$   $k$ -dominates  $R$ . Also, for any given  $A_i$  and any  $v_i \in A_i$ , we know by minimality of  $A_i$  that there is a subset  $S_i$  of  $R$  consisting of  $2k - 2$  vertices which send at most  $k - 1$  nonedges in  $\overline{G}$  into  $A_i - v_i$ . If  $T_i$  is the set of vertices of  $A_i - v_i$  which are not  $k$ -dominated in  $\overline{G}$  by  $S_i$ , then

$$|T_i| \leq \frac{|E_G(S_i, A_i)|}{k - 1} \leq \frac{|S_i|(k - 1)}{k - 1} = |S_i| \leq 2k - 2.$$

Note that  $S_i \cup T_i \cup \{v_i\}$  is a  $k$ -dominating set of  $A_i$  in  $\overline{G}$  with size at most  $4k - 3$ . Finally, as with the  $A_i$ 's, there are sets  $S^* \subseteq R$  and  $T^* \subseteq A^*$ , each

of size at most  $2k - 2$ , such that  $S^* \cup T^*$   $k$ -dominate  $A^*$  in  $\overline{G}$ . Therefore

$$\gamma_k(\overline{G}) \leq |X| + \sum_{i=1}^q (|S_i \cup T_i| + 1) + |S^* \cup T^*| \leq (4k - 3)q + 6k - 5.$$

Putting together the bounds of the last two paragraphs, we have

$$\gamma_k(G)\gamma_k(\overline{G}) \leq 4kd + 16k^2q + \frac{6kd}{q} + 24k^2 \leq 20kd + 30k^2$$

since  $q \leq d/k$  and  $k \leq d$ . Then  $\gamma_k(G)\gamma_k(\overline{G}) \leq kn$  unless  $d \geq n/30$ .

If  $d \geq n/30$ , the sum of the degrees of vertices in  $X$  is at least  $kd$ , so at most  $2k - 2$  vertices of  $X$  have degree at most  $kd/(2k - 1) \geq n/60$  in  $G$ . But since the argument above holds for all  $(2k - 1)$ -sets  $X$ , at most  $2k - 2$  vertices of  $G$  can have degree at most  $kd/(2k - 1)$ . By Theorem 3.3.3, since  $kd/(2k - 1) \geq n/60$ ,  $\gamma_k(G) \leq 60k \log n$ . Since the arguments above are symmetric in  $G$  and  $\overline{G}$ , the same bound holds for  $\gamma_k(\overline{G})$ . But then

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (60k \log n)^2 \leq kn,$$

so we are done. ■

**Lemma 3.3.16** *Suppose  $k \geq 2$  is an integer and  $G$  is a graph of order  $n \geq 72(40k)^{80k}$ . If  $\gamma_k(\overline{G}) \leq 6k - 5$ , then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2). \tag{3.5}$$

*Proof.* Suppose that  $\gamma_k(G)\gamma_k(\overline{G}) \geq (2k - 1)(n - k + 2)$ . By assumption on  $\gamma_k(\overline{G})$ , we have  $\gamma_k(G) \geq (n - k + 1)/3$ .

Let  $M = \{v : d_G(v) < 40k\}$ . By Lemma 3.3.4 with  $q = |M|$ , we have  $|M| \geq n/20$ . Then, by Proposition 3.3.11 the set system  $\mathcal{N}_G$  contains an  $(r, k + 1)$ - $\Delta$ -system  $D = \{N(v_i) : 1 \leq i \leq k + 1\}$  for some  $r < 40k$ . Let  $K$  be the kernel of  $\mathcal{D}$ . As in the proof of Proposition 3.3.12, the vertices  $v_1, \dots, v_{k+1}$   $k$ -dominate  $V(G) - K$  in  $\overline{G}$ . If  $K$   $k$ -dominates all but  $n/10k$

vertices in  $G$ , then  $\gamma_k(G) \leq n/10k + 40k \leq n/6k$ , so  $\gamma_k(G)\gamma_k(\overline{G}) < n$ , a contradiction. Therefore, at least  $n/10k > 2^{40k}(k-1)$  vertices have at most  $k-1$  neighbors in  $G$  in the set  $K$ , so there are  $B \subseteq K$  and  $A \subseteq V(G) - K$  with  $|B| < k$  and  $|A| = k$  such that  $N_G(v) \cap K = B$  for all  $v \in A$ . Then  $A$ ,  $B$ , and the  $v_i$ 's are a  $k$ -dominating set of  $\overline{G}$  with size at most  $3k$ , so we conclude  $\gamma_k(\overline{G}) \leq 3k$ .

If  $k = 2$ , first apply Lemma 3.3.5 to see  $\gamma_2(\overline{G}) \leq 4$ . Then if  $\gamma_2(G)\gamma_2(\overline{G}) > 3(n-1)$ , we know by Lemma 3.3.8 with  $w = 2$  that  $G$  has at least  $(n-9)/12 \geq 8$  vertices of degree at most 1, so  $\mathcal{N}_G$  contains an  $(r, 3)$ - $\Delta$ -system  $\mathcal{A}$  with  $r$  either 0 or 1. By Proposition 3.3.12,  $\gamma_2(\overline{G}) \leq 3$ . If  $\gamma_2(G) \leq n-1$  we are done. If  $\gamma_2(G) = n$ , then  $G$  has two isolated vertices or an isolated edge, so  $\gamma_2(\overline{G}) = 2$ . In either case, the bound holds.

Now suppose that  $k \geq 3$ . If Inequality (3.5) does not hold, then

$$\gamma_k(G) > \frac{(2k-1)(n-k+2)}{3k} \geq \frac{19n}{36} \geq \frac{n}{2} + 2(2k)^{4k-2}.$$

Then, by Lemma 3.3.8 with  $w = 1$ ,  $G$  has at least  $2(2k)^{4k-2}$  vertices of degree less than  $2k-1$ . By Proposition 3.3.11, the set system  $\mathcal{N}_G$  contains an  $(r, k+1)$ - $\Delta$ -system with  $r < 2k-1$ ; let  $\mathcal{A}$  be such a  $\Delta$ -system with minimum  $r$ . Then  $\gamma_k(\overline{G}) \leq r+k$  by Lemma 3.3.12. If  $\gamma_k(G) > wn/(w+1) + n/10k$  with  $w = \lfloor r/(r-k+1) \rfloor$ , then by Lemma 3.3.8  $G$  has at least  $n/10k > 2(2k)^{4k-2}$  vertices of degree less than  $r' = (w+1)k/w - 1$ . From Lemma 3.3.9 we have  $r' < r$ , so  $\mathcal{N}_G$  contains a  $(q, k+1)$ - $\Delta$ -system with  $q \leq r' < r$ , a contradiction. Therefore,

$$\gamma_k(G)\gamma_k(\overline{G}) \leq \left( \frac{wn}{w+1} + \frac{n}{10k} \right) (k+r).$$

If  $r \geq k$ , we can apply Proposition 3.3.10 to see the right hand side is at most

$$n(2k-4/3) + 3kn/10k < (2k-1)(n-k+1).$$

We are left only with the case that  $r < k$ . Let  $K$  be the kernel of  $\mathcal{A}$ . Each vertex of  $K$  has degree at least  $k+1$  in  $G$ , so  $V(G) - K$  is a  $k$ -dominating set

of  $G$ , which tells us  $\gamma_k(G) \leq n - |K|$ . If  $|K| = r$ , Proposition 3.3.12 tells us  $\gamma_k(\overline{G}) \leq |K| + k$ , so  $\gamma_k(G)\gamma_k(\overline{G}) \leq (k + |K|)(n - |K|) \leq (2k - 1)(n - k + 1)$ . If  $|K| < r$ , then Proposition 3.3.12 tells us  $\gamma_k(\overline{G}) \leq k + 1 + |K|$ . Since  $|K|$  must be at most  $k - 2$ ,  $\gamma_k(G)\gamma_k(\overline{G}) \leq (k + 1 + |K|)(n - |K|) \leq (2k - 1)(n - k + 2)$  and the desired bound holds.  $\blacksquare$

**Theorem 3.3.17** *Let  $k \geq 2$  be an integer. Let  $G$  be a graph with order  $n \geq 72(40k)^{80k}$ . Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2).$$

*Proof.* If every  $(2k - 1)$ -set  $X \subseteq V(G)$  satisfies that  $N_G^k(X)$   $k$ -dominates all but at most  $2k - 2$  vertices of  $V(G) - N_G^k[X]$ , then we are done by Lemma 3.3.15 (noting that  $n$  is at least  $10^6 k^2$ ).

Otherwise, let  $X$  be an exceptional  $(2k - 1)$ -set, and let  $Y$  be the vertices of  $V(G) - N_G^k[X]$  not  $k$ -dominated in  $G$  by  $N_G^k(X)$ . Let  $W$  be the set of vertices of  $V(G) - X$  not  $k$ -dominated by  $Y$  in  $\overline{G}$ . Then

$$(|W| - k + 1)|Y| \leq |Y||W| - |E_G(Y, W)| = |E_{\overline{G}}(Y, W)| \leq (k - 1)|W|,$$

so  $|W| \leq |Y|(k - 1)/(|Y| - k + 1) \leq 2k - 2$ . Since  $W \cup X \cup Y$  is a  $k$ -dominating set in  $\overline{G}$ , we have  $\gamma_k(\overline{G}) \leq |W| + |X| + |Y| \leq 6k - 5$ . Now apply Lemma 3.3.16; this gives the Theorem.  $\blacksquare$

### 3.3.6 Nearly Regular Graphs

It may seem that to maximize  $\gamma_k(G)\gamma_k(\overline{G})$ , we would want to make all the degrees of  $G$  small so that  $\gamma_k(G)$  would be large. Perhaps surprisingly, this is not the case. Indeed, if  $\gamma_k(G)\gamma_k(\overline{G})$  is to be large, the maximum degree must be large compared to the minimum degree in either  $G$  or  $\overline{G}$ ; this is the content of Theorem 3.3.18.

**Theorem 3.3.18** *Let  $k \geq 2$  be an integer. Let  $G$  be a graph of order  $n \geq 10^{20}k^{10}(\log k)^{10}$  such that*

$$\Delta(H)\delta(H) \leq \delta(H)^2n^{1/5}$$

*for  $H \in \{G, \overline{G}\}$ . Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (k+1)n.$$

*Furthermore, if  $G$  and  $\overline{G}$  have minimum degree at least  $k$ , then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq kn.$$

Both bounds above are sharp. If  $G$  is a  $(k-1)$ -regular  $K_k$ -free graph with at least  $k^3$  vertices, then we have  $\gamma_k(G) = n$  and  $\gamma_k(\overline{G}) \geq k+1$  (if  $\gamma_k(\overline{G})$  were  $k$ , the vertices of a minimum  $k$ -dominating set in  $\overline{G}$  would form a  $K_k$  in  $G$ ), so  $G$  witnesses the sharpness of the first bound. If we insist  $G$  and  $\overline{G}$  have minimum degree at least  $k$ , we take  $G$  to be a disjoint union of (at least two)  $K_{k+1}$ 's. Then any minimum  $k$ -dominating set of  $G$  contains exactly  $k$  vertices from each component, so  $\gamma_k(G) = kn(G)/(k+1)$ . It is easy to see  $\gamma_k(\overline{G}) = k+1$ , so  $G$  demonstrates the sharpness of the second bound.

*Proof.* By Corollary 3.3.7, without loss of generality,  $\gamma_k(\overline{G}) \leq (2k-1)\lceil \log n \rceil$ .

Let  $M$  denote the set of vertices  $v$  with  $d_G(v) \leq n^{1/5}$ . Suppose  $|M| \leq n^{4/5}$ . Since  $n^{1/5} \geq 7k \log k$ , by Lemma 3.3.3,

$$\gamma_k(G) \leq \left( \frac{k \log n}{5n^{1/5}} + \frac{k(2e \log n)^{k-1}}{5^{k-1}n^{k/5}} \right) n + n^{4/5}.$$

Since  $n$  is large, we have  $2e \log n \leq n^{1/5}$ , so

$$\gamma_k(G) \leq n^{4/5}(k \log n/5 + k/4 + 1) \leq n^{4/5} \log n.$$

But then

$$\gamma_k(G)\gamma_k(\overline{G}) \leq n^{4/5}(k \log n)^2 \leq n.$$

We may therefore assume  $|M| \geq n^{4/5}$ . Since there are at most  $\Delta(G)\delta(G) \leq \delta(G)^2 n^{1/5} \leq n^{3/5}$  vertices of  $M$  within distance 2 in  $G$  from any vertex of  $M$ , we can find a set of  $n^{1/5} \geq k + 1$  vertices pairwise at distance at least 3 in  $G$ . But  $k + 1$  of these vertices together form a  $k$ -dominating set in  $\overline{G}$ , so  $\gamma_k(\overline{G}) \leq k + 1$ . This immediately implies the first bound. If  $\overline{G}$  has minimum degree at least  $k$ , then  $\gamma_k(\overline{G}) \leq kn/(k + 1)$  by Theorem 3.3.1, which gives us the second.  $\blacksquare$

### 3.4 Applications of the Method

In this section we show how to apply the methods of Sections 3.2 and 3.3 to obtain Nordhaus-Gaddum bounds for parameters similar to domination number. More precisely, we establish such bounds for parameters  $P$  for which there are constants  $c_1$  and  $c_2$  such that

$$c_1 \cdot \gamma(G) \leq P(G) \leq c_2 \cdot \gamma(G)$$

for all graphs  $G$ .

We demonstrate the method first in Subsection 3.4.1 on three parameters, namely Roman domination, connected domination, and total domination. These all behave similarly to domination number, so they allow an argument very similar to that in Section 3.2. The *connected domination number* of a connected graph  $G$ , written  $\gamma_c(G)$ , is the minimum size of a dominating set of  $G$  inducing a connected subgraph. The *total domination number* of a graph  $G$  with no isolated vertices, written  $\gamma_t(G)$ , is the minimum size of a dominating set of  $G$  inducing a subgraph with no isolated vertices.

In Subsection 3.4.2, we use results of Section 3.3 with  $k = 2$  to prove a conjecture (Conjecture 3.4.5) of Alon, Balogh, Bollobás, and Szabó [1] on game domination.

### 3.4.1 Domination-Like Parameters

In this subsection we establish Nordhaus-Gaddum bounds on parameters bounded between constant multiples of  $\gamma(G)$ .

The proofs all rely on the following principle:

**Principle:** If a parameter  $P$  is bounded by constant multiples of  $\gamma(G)$ , then  $P(G)P(\overline{G})$  is maximized when  $G$  or  $\overline{G}$  has diameter at least 3.

**Theorem 3.4.1** *Let  $G$  be a graph of order  $n \geq 10777$ . Then*

$$\gamma_R(G)\gamma_R(\overline{G}) \leq \frac{16n}{5}.$$

*Proof.* We know that for all graphs  $G$ ,  $\gamma_R(G) \leq 2\gamma(G)$ . If  $G$  and  $\overline{G}$  have diameter at most 2 then

$$\gamma_R(G)\gamma_R(\overline{G}) \leq 4\gamma(G)\gamma(\overline{G}) \leq 16(n \ln(n))^{2/3}$$

by Theorem 3.2.2. The quantity on the right is at most  $16n/5$  since  $n \geq 10777$ .

We may therefore assume, by symmetry, that  $G$  has diameter at least 3. If  $G$  has an isolated vertex or an isolated edge,  $\gamma_R(\overline{G}) \leq 3$  so  $\gamma_R(G)\gamma_R(\overline{G}) \leq 3n$  and we are done. Otherwise, by a theorem of Chambers, Kinnersley, Prince, and West [16], we have  $\gamma_R(G) \leq 4n/5$ . If  $u$  and  $v$  are at distance at least 3 in  $G$ , then the RDF  $(V(G) - \{u, v\}, \emptyset, \{u, v\})$  of  $\overline{G}$  has weight 4, so  $\gamma_R(\overline{G}) \leq 4$ . But then  $\gamma_R(G)\gamma_R(\overline{G}) \leq 16n/5$ , and we are done. ■

To see that this bound is sharp, consider  $G_k = kC_5$ .  $G_k$  has Roman domination number  $4k$ , and its complement has Roman domination number 4, so

$$\gamma_R(G_k)\gamma_R(\overline{G}_k) = 16k = \frac{16}{5}n(G_k).$$

For connected domination, note if  $G$  is a connected graph, then  $\gamma_c(G) \leq 3\gamma(G) - 2$  (if  $S$  is a dominating set of  $G$  and  $v \in S$ , then the distance from  $v$  to  $S - v$  is at most 3).

**Theorem 3.4.2** *Let  $G$  be a graph of order  $n \geq 1133437$  such that  $G$  and  $\overline{G}$  are connected. Then*

$$\gamma_c(G)\gamma_c(\overline{G}) \leq 2n - 4.$$

*Proof.* Suppose first that  $G$  and  $\overline{G}$  have diameter at most 2. Then by Lemma 3.2.2,

$$\begin{aligned} \gamma_c(G)\gamma_c(\overline{G}) &\leq (3\gamma(G) - 2)(3\gamma(\overline{G}) - 2) \leq 9\gamma(G)\gamma(\overline{G}) \\ &\leq 36(n \ln(n))^{2/3} \leq 2n - 4. \end{aligned}$$

We therefore assume that, by symmetry,  $G$  has diameter at least 3. If  $u$  and  $v$  are at distance at least 3 in  $G$ , then  $\{u, v\}$  is a connected dominating set of  $\overline{G}$ , so  $\gamma_c(\overline{G}) \leq 2$ . Also,  $\gamma_c(G) \leq n - 2$  (the set of vertices of degree at least 2 in any spanning tree of  $G$  is a connected dominating set). Putting this together,  $\gamma_c(G)\gamma_c(\overline{G}) \leq 2n - 4$ . ■

This result is sharp by considering  $G = P_n$ , the path on  $n$  vertices.

We now consider total domination. Notice that for all graphs  $G$  with no isolated vertices,  $\gamma_t(G) \leq 2\gamma(G)$  because we can simply add a neighbor of every vertex in a minimum dominating set to get a total dominating set.

**Theorem 3.4.3** *Let  $G$  be a graph of order  $n \geq 62423$  such that  $G$  and  $\overline{G}$  have no isolated vertices. Then*

$$\gamma_t(G)\gamma_t(\overline{G}) \leq 2n.$$

*Proof.* Suppose first that  $G$  and  $\overline{G}$  have diameter at most 2. Then by Lemma 3.2.2

$$\gamma_t(G)\gamma_t(\overline{G}) \leq 4\gamma(G)\gamma(\overline{G}) \leq 16(n \ln(n))^{2/3} \leq 2n.$$

We therefore assume that, by symmetry,  $G$  has diameter at least 3. If  $u$  and  $v$  are at distance at least 3 in  $G$ , then  $\{u, v\}$  is a total dominating set of

$\overline{G}$ , so  $\gamma_t(\overline{G}) = 2$ . Then trivially  $\gamma_t(G)\gamma_t(\overline{G}) \leq 2n$ , so we are done. ■

The graph  $(n/2)K_2$  shows that Theorem 3.4.3 is sharp.

Theorems 3.4.1, 3.4.2, and 3.4.3 immediately imply the following known bounds for sums (although only for large values of  $n$ ):

$$\gamma_R(G) + \gamma_R(\overline{G}) \leq 3n \quad [16]$$

$$\gamma_c(G) + \gamma_c(\overline{G}) \leq n + 1 \quad [29]$$

$$\gamma_t(G) + \gamma_t(\overline{G}) \leq n + 2 \quad [19].$$

The three bounds above are all sharp.

### 3.4.2 Game Domination

Given a digraph  $D$ , define the *domination number* of  $G$  to be the minimum size of a set  $U \subseteq V(D)$  such that for every vertex  $v \in V(D) - U$ , there is a  $u \in U$  such that  $uv \in E(D)$ .

The game domination number of a graph  $G$  is defined in terms of the following procedure. Two players, D (dominator) and A (avoider or adversary), alternately orient edges of a graph  $G$ , starting with D. The goal of D is to make the domination number of the resulting directed graph as small as possible, whereas A tries to make it large. We define  $\gamma_g(G)$ , the *game domination number of  $G$* , to be the domination number of the resulting digraph if both D and A play their optimal strategy.

Game domination was introduced by Alon, Balogh, Bollobás, and Szabó in [1]. They computed  $\gamma_g$  for several classes of graphs and a number of extremal problems were solved. For example, the following Nordhaus-Gaddum bound was proven.

**Theorem 3.4.4 ([1])** *For any graph  $G$  on  $n$  vertices,  $\gamma_g(G) + \gamma_g(\overline{G}) \leq n + 2$ .*

Theorem 3.4.4 is sharp if  $G = K_n$ . The authors of [1] conjectured this could be improved.

**Conjecture 3.4.5 ([1])** *If  $G$  and  $\overline{G}$  are connected graphs with order  $n$ , then*

$$\gamma_g(G) + \gamma_g(\overline{G}) \leq 2n/3 + 3.$$

In the remainder of this section, we prove Conjecture 3.4.5 if  $n$  is large.

We first recall some useful results.

**Theorem 3.4.6 ([1])** *If  $G$  is a connected graph on  $n \geq 2$  vertices, then  $\gamma_g(G) \leq 2n/3$ . Furthermore, if  $G$  has no isolated vertices, then also  $\gamma_g(G) \leq 2n/3$ . If  $G$  is a graph with minimum degree at least 2, then  $\gamma_g(G) \leq n/2$ .*

Suppose D and A are playing their domination game on a graph  $G$ , and let  $X$  be a 2-dominating set of  $G$ . As soon as A orients an edge from  $v \in V(G) - X$  into  $X$ , D, on his next turn, can orient an edge from  $X$  into  $v$ . In this way,  $X$  dominates the oriented digraph D and A create. Therefore, as noted in [1],  $\gamma_g(G) \leq \gamma_2(G)$  for all graphs  $G$ .

We can sometimes do better than this, though. Define a *nearly-2-dominating set* of  $G$  to be a set  $X \subseteq V(G)$  such that  $X$  dominates  $G$  and 2-dominates all but at most one vertex of  $G$ . Let  $\gamma_2^*(G)$  be the size of the smallest nearly-2-dominating set of  $G$ . Then, if  $X$  is a nearly-2-dominating set in  $G$  and  $v$  is the vertex not 2-dominated by  $X$  (if it exists), then D can orient an edge from  $X$  to  $v$  on his first turn, and then orient edges from  $X$  to the other vertices of  $V(G) - X$  in response to A. This observation gives our next Proposition.

**Proposition 3.4.7** *For all graphs  $G$ ,  $\gamma_g(G) \leq \gamma_2^*(G) \leq \gamma_2(G)$ .*

This Proposition is the bridge from our earlier results to game domination.

**Theorem 3.4.8** *For any  $n$ -vertex graph  $G$  with  $n \geq 4 \cdot 10^6$ ,*

$$\gamma_g(G)\gamma_g(\overline{G}) \leq 2n + 12.$$

*Proof.* By Theorem 3.3.15 and Proposition 3.4.7, we are done unless (without loss of generality) there is some 3-set  $X$  such that  $N_G^2(X)$  does not 2-dominate two vertices  $v_1$  and  $v_2$ . Let  $y_i$  be the neighbor in  $G$  of  $v_i$  in the set  $N_G^2(X)$  for  $i = 1, 2$ , if it exists. Then  $X \cup \{v_1, v_2, y_1, y_2\}$  is a 2-dominating set of  $\overline{G}$ , so  $\gamma_2(\overline{G}) \leq 7$ .

By Lemma 3.3.5 and Proposition 3.4.7, we may assume that  $\gamma_2(\overline{G}) \leq 4$ .

Suppose  $G$  has three vertices of degree at most 1, and let  $v_1, v_2$ , and  $v_3$  be the vertices of smallest degree in  $G$ . Let  $x_i$  be the neighbor of  $v_i$  (if it exists) for  $i \in \{1, 2, 3\}$ . Then  $\{v_1, v_2, v_3\}$  is a 2-dominating set of  $\overline{G}$  unless, without loss of generality,  $y_1 = y_2$ . In that case,  $\{v_1, v_2, y_1\}$  is a 2-dominating set of  $\overline{G}$ . In either case,  $\gamma_g(\overline{G}) \leq 3$ . By Theorem 3.4.6, if  $G$  has no isolated vertex,  $\gamma_g(G) \leq 2n/3$ , implying  $\gamma_g(G)\gamma_g(\overline{G}) \leq 2n$ . Suppose, then, that  $v_1$  is isolated. Then  $\{v_1, v_2\}$  is a nearly-2-dominating set of  $\overline{G}$ , so  $\gamma_g(G)\gamma_g(\overline{G}) \leq 2\gamma_g(G) \leq 2n$ .

Therefore, we may assume that  $G$ , and by symmetry  $\overline{G}$ , has at most two vertices of degree at most 1. It follows from Theorem 3.4.6 that  $\gamma_g(G)$  and  $\gamma_g(\overline{G})$  are each at most  $n/2 + 3$  (add at most 3 edges to  $G$  which are incident to at most 4 vertices to increase the minimum degree to 2, then consider deleting those edges). From the second paragraph, one of these should also be at most 4. Therefore  $\gamma_g(G)\gamma_g(\overline{G}) \leq 2n + 12$ , so we are done. ■

**Theorem 3.4.9** *Let  $G$  be a graph with  $n \geq 4 \cdot 10^6$  vertices such that  $G$  and  $\overline{G}$  are connected. Then  $\gamma_g(G) + \gamma_g(\overline{G}) \leq 2n/3 + 3$ . That is, Conjecture 3.4.5 holds for graphs of order at least  $4 \cdot 10^6$ .*

*Proof.* Maximize  $x + y$  ( $x = \gamma_g(G)$ ,  $y = \gamma_g(\overline{G})$ ) subject to  $x, y \leq 2n/3$  (by Theorem 3.4.6) and  $xy \leq 2n + 12$  (by Theorem 3.4.8). ■

Notice Theorem 3.4.8 can also be used to prove Theorem 3.4.4.

### 3.5 Thresholds for Roman Domination

The Roman domination number of a graph  $G$ ,  $\gamma_R(G)$ , is a variant of domination number introduced in [20] and studied further in [21, 16, 30]. A Roman dominating function (RDF) of  $G$  is a function  $f : V(G) \rightarrow \{0, 1, 2\}$  such that every vertex in  $f^{-1}(0)$  has a neighbor in  $f^{-1}(2)$ . The weight of  $f$ ,  $w(f)$ , is  $\sum_{v \in V(G)} f(v)$ . By definition,  $\gamma_R(G)$  is the minimum weight of an RDF of  $G$ . For an RDF  $f$  and  $i \in \{0, 1, 2\}$ , we will write  $V_i(f)$  for  $f^{-1}(i)$ . We suppress the dependence on  $f$  when the context is clear.

In some ways, the quantity  $\gamma_R(G)/\gamma(G)$  can be considered a measure of the efficiency of domination in  $G$ . When  $\gamma_R(G)/\gamma(G)$  is close to 1, most vertices (namely, those in  $V_1$ ) in the near-optimal dominating set  $V_1 \cup V_2$  are only needed to dominate themselves, which seems inefficient. On the other hand, when  $\gamma_R(G)/\gamma(G)$  is close to 2, most of the vertices (those given weight 2) in a minimum dominating set of  $G$  are needed to dominate other vertices as well, which is much more economical.

The goal of this section is to determine when these ratios are achieved (asymptotically) for a random graph in  $\mathbb{G}_{n,p}$ .

Although domination and random graphs have each been studied extensively, domination in random graphs has received very little attention. Wieland and Godbole [49] essentially determined the domination number for  $\mathbb{G}_{n,p}$  when  $p$  is not too small, demonstrating a two-point concentration. For small  $p$ , though, little is known. While we do not have quantitative information, Theorem 3.5.1 does give a sort of qualitative description of domination in  $\mathbb{G}_{n,p}$  when  $p$  is very close to 0.

Throughout the section,  $\mathbb{G}_{n,p}$  denotes the probability space of all graphs on  $n$  labelled vertices, where each of the  $\binom{n}{2}$  edges appears randomly and independently with probability  $p = p(n)$ . We use  $\mathbb{P}$  and  $\mathbb{E}$  to denote probability and expectation. Furthermore, for  $A \subset \mathbb{G}_{n,p}$ ,  $\mathbb{E}_A(X)$  is the expectation of  $X$  over the set  $A$ . If  $\mathbb{P}(E) \rightarrow 1$  as  $n \rightarrow \infty$ , we say the event  $E$  happens

with high probability (alternatively “almost always” or “for almost every  $G$ ”), abbreviated w.h.p. For two functions  $f(n)$  and  $g(n)$ , we say  $f \sim g$  if  $f/g$  tends to 1 as  $n$  tends to  $\infty$ . We use  $o(g)$  to denote a function  $f$  such that  $f/g \rightarrow 0$  as  $n \rightarrow \infty$ . As conventions,  $q = 1 - p$  and  $w = pn$ .

Corollary 3.5.5 will tell us that  $\gamma_R(G) \geq 2\gamma(G) - 1$  if  $G$  has diameter 2. It is known ([11], Corollary 10.11) that if  $p^2n - 2 \log n \rightarrow \infty$ , then almost every  $G \in \mathbb{G}_{n,p}$  has diameter 2. Therefore the probability threshold for the event  $\gamma_R(G) = (2 - o(1))\gamma(G)$  should be no larger than  $\sqrt{2 \log n/n}$ ; the actual threshold is in fact much lower.

As it turns out, the number of isolated vertices guides the ratio  $\gamma_R/\gamma$ : when there are many isolates ( $n - o(n)$ ), the ratio is nearly 1, and when there are few isolates ( $o(n)$ ), the ratio is close to 2. This constitutes our main result.

**Theorem 3.5.1** *Let  $p : \mathbb{N} \rightarrow \mathbb{R}$  be a function satisfying  $p \leq 1/2$ . Let  $w = pn$ . Then for almost every  $G \in \mathbb{G}_{n,p}$ ,*

- (1) *if  $w \rightarrow 0$ , then  $\gamma_R(G) = (1 + o(1))\gamma(G)$*
- (2) *if  $w \rightarrow \infty$ , then  $\gamma_R(G) = (2 - o(1))\gamma(G)$*
- (3) *if  $w \rightarrow c \in \mathbb{R}$ , then there is a constant  $c' \in (1, 2)$  such that  $\gamma_R(G) = (c' + o(1))\gamma(G)$ .*

The reason we insist  $p \leq 1/2$  in Theorem 3.5.1 is that  $p$  must be bounded away from 1 to ensure that  $\gamma(G)$  is (almost always) unbounded, and  $1/2$  is simply convenient. The Theorem would remain true if we stipulated that  $p \leq b$  for any  $b < 1$ , or even  $p \rightarrow 1$  slowly. The only modification needed would be in the bound on  $q^{\alpha n}$  in the second paragraph of the Proof of Theorem 3.5.1. However, based on the discussion above about graphs of diameter 2, we are justified in requiring  $p \rightarrow 0$ ; we will therefore assume this through the rest of the section.

The Section is organized as follows: in Subsection 3.5.1, we develop lemmas we will need for the proof of Theorem 3.5.1, which follows in Subsection 3.5.2.

### 3.5.1 Some Lemmas

In this subsection we prove several lemmas we will need in the next subsection. We begin with two on degrees in random graphs.

In the remainder of the section, let  $X_k(G)$  be the random variable counting the vertices of degree  $k$  in  $G$ . A simple first moment/second moment argument shows:

**Lemma 3.5.2** *If  $p \rightarrow 0$  and  $w = pn$  then, for almost every  $G \in \mathbb{G}_{n,p}$ ,*

$$X_0(G) \sim \mathbb{E}(X_0) \sim \frac{n}{e^w}.$$

The next lemma tells when very few vertices have degree significantly less than the expected degree ( $\approx w$ ).

**Lemma 3.5.3** *Suppose  $p \rightarrow 0$  and  $w \rightarrow \infty$ . Then almost every  $G \in \mathbb{G}_{n,p}$  has at most  $(9/10)^w n$  vertices of degree less than  $w/2$ .*

*Proof.* Choose  $n$  large enough that  $p \leq 1/100$ . Now, the number of vertices of degree less than  $k$  is  $\sum_{i=0}^{k-1} X_i$ ; call this quantity  $Y_k$ . Then

$$\begin{aligned} \mathbb{E}(Y_{w/2}) &= n \sum_{i=0}^{w/2-1} \binom{n}{i} p^i (1-p)^{n-1-i} \leq n \sum_{i=0}^{w/2-1} \frac{n^i p^i}{i!} e^{p(n-i-1)} \\ &= n \sum_{i=0}^{w/2-1} \frac{w^i e^{p(i+1)}}{i! e^w} \leq (1+o(1)) n \frac{w}{2} \frac{w^{w/2} e^{101w/200}}{(w/2)^{w/2} e^w} \\ &< \frac{(1+o(1)) w (.863)^w}{2} n < .87^w n \end{aligned}$$

if  $n$  is large enough. The Markov bound says  $\mathbb{P}(X \geq m) \leq \mathbb{E}(X)/m$ , so the probability that  $G$  has at least  $(9/10)^w n$  vertices of degree less than  $w/2$  is at most  $(.87/.9)^w \rightarrow 0$ . ■

We now turn our attention to Roman domination. In particular, we try to relate  $\gamma_R(G)$  to properties of the degrees of vertices of  $G$ . Our first lemma deals with optimal RDFs; its corollaries will prove quite useful.

**Lemma 3.5.4** *Let  $G$  be a graph. Then  $G$  has a minimum weight RDF such that*

- 1)  $V_1$  is an independent set,
- 2) each vertex in  $V_0$  has at most 1 neighbor in  $V_1$ , and
- 3)  $E_G[V_1, V_2] = \emptyset$ .

*Proof.* Let  $f$  be any minimum-weight RDF of  $G$ . If  $G[V_1]$  contains edge  $uv$ , move  $u$  to  $V_2$  and  $v$  to  $V_0$ . If some vertex  $u \in V_0$  has neighbors  $v$  and  $w$  in  $V_1$ , move  $u$  to  $V_2$  and  $v$  and  $w$  to  $V_0$ . If  $u \in V_1$ ,  $v \in V_2$ , and  $uv \in E(G)$ , then move  $u$  to  $V_0$ . None of these moves increase the weight of  $f$ . Since each move decreases  $|V_1|$ , only a finite sequence of moves is possible. When no more such move is left, we have the claimed RDF. ■

**Corollary 3.5.5** *Let  $G$  be a graph of diameter 2. Then  $\gamma_R(G) \geq 2\gamma(G) - 1$ .*

*Proof.* Let  $f$  be a minimum weight RDF of  $G$  as claimed in Lemma 3.5.4. Suppose  $|V_1| \geq 2$ , and let  $u, v \in V_1$  ( $u \neq v$ ). Since  $G$  has diameter 2,  $u$  and  $v$  have a common neighbor  $w$ . Now  $w \notin V_1$  by (1),  $w \notin V_0$  by (2), and  $w \notin V_2$  by (3). Therefore,  $|V_1| \leq 1$ . Since  $|V_1 \cup V_2| \geq \gamma(G)$ ,  $w(f) = |V_1| + 2|V_2| \geq 2\gamma(G) - 1$ . ■

**Corollary 3.5.6** *Let  $G$  be a graph with at most  $m$  vertices of degree less than  $k \geq 1$ . Then*

$$\gamma_R(G) \geq 2\gamma(G) - \frac{n}{k+1} - m.$$

*Proof.* Let  $f$  be a minimum weight RDF of  $G$  guaranteed by Lemma 3.5.4. There are at least  $|V_1| - m$  vertices of  $V_1$  with degree at least  $k$ , and since their neighborhoods are disjoint and contained in  $V_0$ , we see

$$k(|V_1| - m) \leq |V_0| \leq n - |V_1|,$$

and so  $|V_1| \leq n/(k+1)+m$ . Since  $|V_1 \cup V_2| \geq \gamma(G)$  and  $\gamma_R(G) = 2|V_2| + |V_1| \geq 2\gamma(G) - |V_1|$ , the result is immediate.  $\blacksquare$

### 3.5.2 The Main Result

We are now ready to prove Theorem 3.5.1.

*Proof of Theorem 3.5.1.* Suppose  $w \rightarrow 0$ . By Lemma 3.5.2, w.h.p.,  $X_0 \sim n$ . All vertices of  $X_0$  are in any dominating set, so almost always

$$(1 - o(1))n = X_0 \leq \gamma(G) \leq \gamma_R(G) \leq n,$$

showing that  $\gamma_R(G) = (1 + o(1))\gamma(G)$ . This establishes (1).

Now consider  $w \rightarrow \infty$ . Let  $\alpha = \ln \ln w/w$ . Define  $\Gamma_r$  to be the number of dominating sets of  $G \in \mathbb{G}_{n,p}$  with cardinality  $r$ . Then

$$\begin{aligned} \mathbb{E}(\Gamma_{\alpha n}) &= \binom{n}{\alpha n} (1 - q^{\alpha n})^{n - \alpha n} \\ &\sim \left( \frac{(1 - q^{\alpha n})^{1 - \alpha}}{\alpha^\alpha (1 - \alpha)^{1 - \alpha}} \right)^n \frac{1 + o(1)}{\sqrt{2\pi\alpha(1 - \alpha)n}}. \end{aligned}$$

Since  $\left( \sqrt{2\pi\alpha(1 - \alpha)n} \right)^{-1} = o(1)$ ,  $\mathbb{E}(\Gamma_{\alpha n}) \rightarrow 0$  if

$$(1 - q^{\alpha n})^{1 - \alpha} < \alpha^\alpha (1 - \alpha)^{1 - \alpha}$$

when  $n$  is large. Since  $p < 1/2$ , we have  $q \geq e^{-2p}$ , and so

$$1 - q^{\alpha n} \leq 1 - e^{-2p\alpha n} \leq \exp(-e^{-2p\alpha n}).$$

Hence we need only show

$$\exp((1 + o(1))(-e^{-2p\alpha n})) < \alpha^\alpha (1 - \alpha)^{1 - \alpha}.$$

Taking reciprocals and logarithms and rewriting in terms of  $w$ , the LHS is asymptotically  $(\ln w)^2$ , while the RHS is

$$\frac{\ln \ln w}{w} \ln \left( \frac{w}{\ln \ln w} \right) + \frac{w - \ln \ln w}{w} \ln \left( \frac{w}{w - \ln \ln w} \right) < \frac{\ln(w) \ln \ln(w)}{w}$$

when  $w$  is large enough. We conclude that  $\mathbb{E}(\Gamma_{\alpha n}) \rightarrow 0$ , and that w.h.p.  $\gamma(G) \geq (\ln \ln w/w)n$ .

By Lemma 3.5.3, almost every  $G$  has at most  $(9/10)^w n$  vertices of degree less than  $w/2$ . Therefore we can apply Lemma 3.5.6 to see

$$\gamma_R(G) \geq 2\gamma(G) - (2/w + (9/10)^w) n.$$

From the last paragraph we know that for almost every  $G$ ,  $(2/w + (9/10)^w)n = o(\gamma(G))$ , which establishes (2).

Finally, for (3), suppose  $w \rightarrow c \in \mathbb{R}$  and let  $\mathcal{A} \subset \mathbb{G}_{n,p}$  be the set of graphs with a vertex of degree at least  $K = \ln \ln n$ . Then

$$\mathbb{P}(\mathcal{A}) \leq n \binom{n}{K} p^K \leq \frac{nw^K}{K!} \rightarrow 0.$$

Let  $D_0, \dots, D_n$  and  $RD_0, \dots, RD_n$  be the vertex-exposure martingales given by

$$D_i = \mathbb{E}_{\bar{\mathcal{A}}}(\gamma(G) | G[v_1, \dots, v_i])$$

$$RD_i = \mathbb{E}_{\bar{\mathcal{A}}}(\gamma_R(G) | G[v_1, \dots, v_i]).$$

On  $\bar{\mathcal{A}}$ ,  $|D_{i+1} - D_i| \leq K$  and  $|RD_{i+1} - RD_i| \leq 2K$ . Azuma's Inequality says

$$\mathbb{P}(|X_n - X_0| \geq \lambda\sqrt{n}) \leq e^{-\lambda^2/2},$$

so

$$\mathbb{P}_{\bar{\mathcal{A}}}\left(D_0 - \sqrt{n \ln^3 n} \leq \gamma(G) \leq D_0 + \sqrt{n \ln^3 n}\right) \geq 1 - \frac{1}{\sqrt{n}}$$

$$\mathbb{P}_{\bar{\mathcal{A}}}\left(RD_0 - 2\sqrt{n \ln^3 n} \leq \gamma_R(G) \leq RD_0 + 2\sqrt{n \ln^3 n}\right) \geq 1 - \frac{1}{\sqrt{n}}.$$

Since  $\gamma(G)$  and  $\gamma_R(G)$  are at least  $n/\ln \ln n$  on  $\overline{\mathcal{A}}$ , we see that w.h.p.  $\gamma(G) \sim D_0$  and  $\gamma_R(G) \sim RD_0$ . Let  $c' = RD_0/D_0$ . To establish (3), it remains only to show  $c' \in (1, 2)$ .

By Lemma 3.5.2,  $X_0 \sim n/e^w$ , so

$$\gamma_R(G) \leq 2\gamma(G) - \frac{n + o(n)}{e^w} \leq \left(2 - \frac{1 + o(1)}{e^w}\right) \gamma(G),$$

and  $c' < 2$ . Let  $Z$  be the number of triples  $x, y, z \in V(G)$  such that  $x$  and  $z$  are adjacent to  $y$  and no other edges contain  $x, y$ , or  $z$ . A simple calculation shows  $\mathbb{E}(Z) = (1 + o(1))nw^2/2e^{3w}$  and  $\mathbb{E}(Z^2) = (1 + o(1))\mathbb{E}(Z)$ , so by the Chernoff Bound,  $Z \sim nw^2/2e^{3w}$ . Any optimal weight RDF puts weight 2 on the central vertex of each triple counted by  $Z$ , so  $|V_2| \geq Z$ , and

$$\gamma_R(G) \geq \gamma(G) + Z \geq \gamma(G) + \frac{(1 + o(1))w^2}{2e^{3w}}n \geq \left(1 + \frac{(1 + o(1))w^2}{2e^{3w}}\right) \gamma(G),$$

so  $c' > 1$ . This completes the proof.  $\square$

# Chapter 4

## Minimum Difference Representations

### 4.1 Introduction

A representation of a graph  $G$  is an assignment of sets of a given type (intervals, convex sets, subsets of  $\mathbb{N}$ , etc.) to the vertices of  $G$  such that  $u, v \in V(G)$  are adjacent if and only if  $S(u)$  and  $S(v)$  satisfy a given condition (are disjoint, intersect, etc.). Many different kinds of graph representations have been studied extensively (see, for example, [9], [17], or [37]).

In [13], Boros, Gurvich, and Meshulam defined three new types of graph representations. With each vertex  $v$  of  $G$ , a finite set  $S(v)$  is associated. In the first model,  $u$  and  $v$  are adjacent if and only if

$$(|S(u) - S(v)| + |S(v) - S(u)|) / 2 \geq k,$$

in the second, if and only if

$$\max\{|S(u) - S(v)|, |S(v) - S(u)|\} \geq k,$$

and in the third, if and only if

$$\min\{|S(u) - S(v)|, |S(v) - S(u)|\} \geq k.$$

Let  $\rho_{\text{avg}}(G)$ ,  $\rho_{\text{max}}(G)$ , and  $\rho_{\text{min}}(G)$  be the smallest  $k$  for which  $G$  has such a representation in the first, second, and third model, respectively. In [13] it was shown that each of these parameters exists and is at most  $n - 1$ , where  $n$  is the order of the graph  $G$ . It was also shown that the first two parameters could be large, namely, at least  $\Omega(\log n)$ . Füredi [26] showed that, for almost all  $n$ -vertex graphs, all three parameters are at most  $O(n/\log n)$  and  $\rho_{\text{max}}(G) = \Omega(n/\log n)$ . However, no non-trivial lower bounds on  $\rho_{\text{min}}(G)$  are known, and in [13] it was noted that no graphs were known for which  $\rho_{\text{min}}(G) > 2$ .

In this chapter, we will show that  $\{\rho_{\text{min}}(G)\}$  is unbounded by demonstrating that, for any  $k \in \mathbb{N}$ , almost every ‘large’ graph  $G$  satisfies  $\rho_{\text{min}}(G) > k$ .

**Theorem 4.1.1** *Let  $k$  be a positive integer. Then  $G \in \mathbb{G}_{n,1/2}$  satisfies  $\rho_{\text{min}}(G) > k$  with high probability as  $n \rightarrow \infty$ .*

Our method is as follows. First, we establish a Ramsey-type result on traces of hypergraphs (Theorem 4.2.2) saying that a large enough family of sets must contain one of four configurations. When we apply this to the sets of an optimal  $k$ -minimum-difference-representation ( $k$ -MDR) of  $G$ , that is,  $\{S(v) : v \in V(G)\}$ , we can immediately rule out three of these (see Lemma 4.3.3). The remainder of the proof is dedicated to ruling out the fourth.

Throughout the chapter, all logarithms will have base 2.

After proving Theorem 4.1.1, Balogh and Prince learned that Füredi [26] actually showed the stronger result that for almost all graphs  $G \in \mathbb{G}_{n,1/2}$ ,  $\rho_{\text{min}}(G) = \Omega(\log n)$ .

## 4.2 Traces

In this section we prove a Ramsey-type theorem on set systems. Define the *trace* of a set system  $\mathcal{H}$  on  $X$  to be the system  $\mathcal{H}_X = \{E \cap X : E \in \mathcal{H}\}$ . We

say that  $\mathcal{H}_X$  is *induced* by  $X$ , and the *trace sets* are the sets  $E \cap X$ . We say  $E$  is an *ancestor in  $\mathcal{H}$*  of  $E \cap X \in E(H_X)$ . We also consider any subsystem of  $\mathcal{H}_X$  to be a trace of  $\mathcal{H}$  induced by  $X$ .

Define an  $(\ell, k)$ -*star* to be a set system  $\{A_i : 1 \leq i \leq \ell\}$  whose elements are pairwise disjoint and all have cardinality  $k$ . An  $(\ell, k)$ -*costar* is a set system  $\{A_i : 1 \leq i \leq \ell\}$  where, if  $A = \bigcup_i A_i$ , then  $\{A - A_i : 1 \leq i \leq \ell\}$  is an  $(\ell, k)$ -star. We refer to an  $(\ell, 1)$ -star or  $(\ell, 1)$ -costar simply as an  $\ell$ -star or  $\ell$ -costar, respectively. An  $m$ -*double-chain* is a system  $\{\{v_i, \dots, v_{i+m-1}\} : 1 \leq i \leq m\}$ .

Balogh and Bollobás [5] proved the following result on traces of set systems.

**Theorem 4.2.1** *Let  $\ell$  and  $m$  be positive integers and let  $g(\ell, m) = (2m)^\ell 2^{9\ell}$ . Then any antichain  $\mathcal{H}$  with  $|\mathcal{H}| \geq g(\ell, m)$  contains either an  $\ell$ -star, an  $\ell$ -costar, or an  $m$ -double-chain as a trace.*

A short proof of Theorem 4.2.1, along with other similar results, appeared in [8]. Recently, this result has seen many applications, for example in [4], [6], and [7].

Our new theorem includes a fourth kind of set system. Define an  $(r, p, q)$ -*system* to be a set system

$$\{K \cup F_i \cup \bigcup_{j \neq i} G_j \cup \bigcup_{j \leq i} L_j : 1 \leq i \leq r\}$$

such that

- (1)  $K$  is the *kernel* of the system,
- (2)  $K$  and the  $3r$  sets  $F_i$ ,  $G_i$ , and  $L_i$  for  $1 \leq i \leq r$  are pairwise disjoint,
- (3)  $|F_i| = p$  and  $|G_i| = q$  for all  $1 \leq i \leq r$ .

Let  $g$  be the function of Theorem 4.2.1. Define  $h_1(\ell, m, r) = g(\ell, m)$  and  $h_k(\ell, m, r) = g(r^2 \cdot h_{k-1}(\ell, m, r), m)$  for  $k > 1$ .

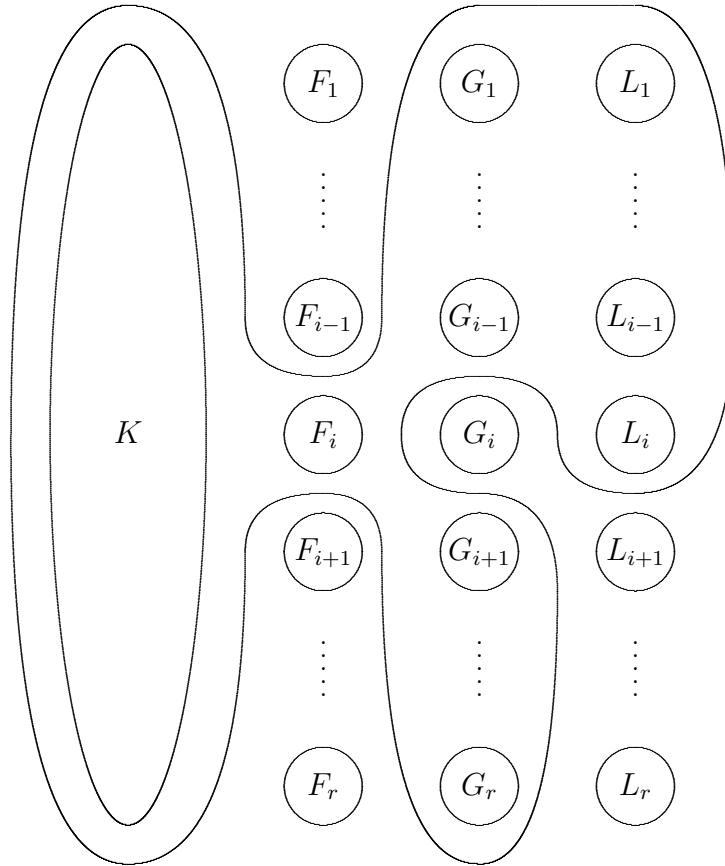


Figure 4.1: A general element of an  $(r, p, q)$ -system

**Theorem 4.2.2** *Let  $\ell$ ,  $k$ ,  $m$ , and  $r$  be positive integers. Then every antichain  $\mathcal{H}$  with  $|\mathcal{H}| \geq h_{2k}(\ell, m, r)$  contains an  $(\ell, k)$ -star, an  $(\ell, k)$ -costar, or an  $m$ -double-chain as a trace or contains an  $(r, p, q)$ -system with  $p, q \leq k$  as a subsystem. Furthermore, we may insist that the  $(r, p, q)$ -system have  $L_i = \emptyset$  for all  $i$  or  $L_1 = \emptyset$  and  $L_i \neq \emptyset$  for all  $i \geq 2$ .*

Throughout the proof, we use the term *collection* to mean a set whose elements may occur with multiplicity greater than 1.

*Proof.* If  $\mathcal{H}$  has an  $m$ -double-chain as a trace, we are done. Otherwise, implement the following algorithm:

**Step 1:** Initialize  $p = q = 0$ ,  $i = 1$  and  $\mathcal{H}_1 = \mathcal{H}$ .

**Step 2:** Let  $d_i = r^2 \cdot h_{2k-i}(\ell, m, r)$ . If  $\mathcal{H}_i$  contains no  $d_i$ -star trace, move to Step 3. Otherwise, let  $T_1^i, \dots, T_{d_i}^i$  be the trace sets of a  $d_i$ -star trace, increment  $p$  by 1, and call the set inducing the star  $B_p$ . Let  $A_j^i$  be an ancestor in  $\mathcal{H}_i$  of  $T_j^i$ , and define  $\mathcal{H}_{i+1}$  to be the set system  $\{A_j^i - B_p : 1 \leq j \leq d_i\}$ . Increment  $i$  by 1. If  $p = k$ , stop and output the ancestors in  $\mathcal{H}$  of the  $T_j^i$ 's; these sets contain an  $(\ell, k)$ -star trace. Otherwise repeat Step 2.

**Step 3:** If  $\mathcal{H}_i$  contains no  $d_i$ -costar trace, move to Step 4. Otherwise, let  $T_1^i, \dots, T_{d_i}^i$  be the trace sets of a  $d_i$ -costar trace, increment  $q$  by 1, and call the set inducing the costar  $C_q$ . Let  $A_j^i$  be an ancestor in  $\mathcal{H}_i$  of  $T_j^i$ , and define  $\mathcal{H}_{i+1} = \{A_j^i - C_q : 1 \leq j \leq d_i\}$ . Increment  $i$  by 1. If  $q = k$ , stop and output the ancestors in  $\mathcal{H}$  of the  $T_j^i$ 's; these sets contain an  $(\ell, k)$ -costar trace. Otherwise return to Step 2.

**Step 4:** Find  $D_1, D_2, \dots, D_r$  among the sets  $T_1^i, \dots, T_{d_i}^i$  such that  $D_1 = D_2 = \dots = D_r$  or  $D_1 \subset D_2 \subset \dots \subset D_r$ , and let  $\mathcal{D} \subset \mathcal{H}$  be the collection of ancestors in  $\mathcal{H}$  of the  $D_i$ 's. Stop and output  $\mathcal{D}$ ; these sets contain an  $(r, p, q)$ -system with  $p, q \leq k$ .

We claim this algorithm outputs an  $(\ell, k)$ -star, an  $(\ell, k)$ -costar, or an  $(r, p, q)$ -system with  $p, q \leq k$ .

First, we show the algorithm is well-defined through  $2k - 1$  steps. Since  $i \leq 2k - 1$ ,  $d_i$  is well-defined, and so Steps 2 and 3 are valid. Therefore, the only thing that must be checked is that when we enter Step 4, we can always find  $r$  sets all equal or ordered linearly by proper inclusion. Suppose that we enter Step 4 on the  $i$ th turn. If any set appears with multiplicity  $r$  among the  $T_j^i$ 's, then we can take  $\mathcal{D}$  to be  $r$  distinct ancestors in  $\mathcal{H}$  of this set. Otherwise, there are at least  $d_i/r = r \cdot h_{2k-i}(\ell, m, r)$  distinct sets among the  $T_j^i$ 's, so by Dilworth's Theorem there is a chain of length  $r$  or an antichain of size  $h_{2k-i}(\ell, m, r)$  among the  $T_j^i$ 's. In the first case we are clearly done. In the second, this antichain contains a  $d_i$ -star, a  $d_i$ -costar, or an  $m$ -double-chain as a trace by Theorem 4.2.1. The first two cases are impossible since we did not complete Steps 2 and 3, and the third is impossible by assumption. Hence such  $D_1, D_2, \dots, D_r$  exist.

Second, the algorithm always terminates with  $i \leq 2k - 1$  since  $p + q$  increases by 1 after every complete implementation of Step 2 or 3 and we either stop when  $\max(p, q) = k$ , or we stop after the first implementation of Step 4.

Finally, we show the algorithm produces what we want. If the algorithm terminates during the  $i$ th turn because  $p = k$ , then we found a star trace  $T_1^i, \dots, T_{d_i}^i$  where  $d_i \geq \ell$ . Note that the  $T_j^i \cap B_n$ 's are disjoint and have cardinality 1 for all  $1 \leq j \leq d_i$  and  $1 \leq n \leq p = k$ . Letting  $A_j^i$  be an ancestor in  $\mathcal{H}$  of  $T_j^i$ , we see  $A_1^i, \dots, A_\ell^i$  form an  $(\ell, k)$ -star induced by

$$\left( \bigcup_{n=1}^p B_n \right) \cap \left( \bigcup_{i=1}^{\ell} A_j^i \right).$$

The case when the algorithm terminates because  $q = k$  is similar; we have

an  $(\ell, k)$ -costar induced by

$$\left( \bigcup_{n=1}^q C_n \right) \cap \left( \bigcup_{i=1}^{\ell} A_j^i \right).$$

Now, if the algorithm terminated after an implementation of Step 4, let  $A_1, \dots, A_r$  be distinct such that  $A_i$  is an ancestor in  $\mathcal{H}$  of  $D_i$  for  $1 \leq i \leq r$ . Then the sets  $A_i \cap B_n$  and  $C_s - A_i$  are pairwise disjoint and have cardinality 1 for all  $1 \leq i \leq r$ ,  $1 \leq n \leq p$ , and  $1 \leq s \leq q$ . Let  $F_i = \bigcup_{n=1}^p (A_i \cap B_n)$ ,  $G_i = \bigcup_{s=1}^q (C_s - A_i)$ ,  $L_1 = \emptyset$  and  $L_i = D_i - D_{i-1}$  for  $2 \leq i \leq r$ , and let  $K = D_1 \cup (\bigcup_{n=1}^q C_n - \bigcup_{i=1}^r A_i)$ . Then we have, for all  $i$ , that

$$A_i = K \cup F_i \cup \bigcup_{j \neq i} G_j \cup \bigcup_{j \leq i} L_j,$$

and  $0 \leq |F_i| = p \leq k$  and  $0 \leq |G_i| = q \leq k$ , so the  $A_i$ 's form an  $(r, p, q)$ -system with kernel  $K$ . Note that if the  $D_i$ 's were all the same, then  $L_i = \emptyset$  for all  $i$ , whereas if they were all different,  $L_1$  is empty and  $L_i \neq \emptyset$  for all  $i \geq 2$ . ■

### 4.3 The Main Result

In this section, we prove our main result, namely, Theorem 4.1.1. Before we begin, we must recall two basic facts about random graphs.

**Fact 4.3.1** *Almost every  $G \in \mathbb{G}_{n,1/2}$  has an independent set on  $\log n$  vertices.*

**Fact 4.3.2** *Let  $G \in \mathbb{G}_{n,1/2}$  and let  $S \subseteq V(G)$  with  $|S| = o(\log n)$ . If  $S$  is the disjoint union of  $A$  and  $B$ , then with high probability there is a vertex  $v$  such that  $v$  is adjacent to every vertex in  $A$  and to no vertex in  $B$ .*

We begin with an easy observation.

**Lemma 4.3.3** *Let  $\mathcal{H}$  be a set system satisfying  $\min\{|S_i - S_j|, |S_j - S_i|\} < k$  for all  $S_i, S_j \in \mathcal{H}$ . Then  $\mathcal{H}$  has no  $(2, k)$ -star,  $(2, k)$ -costar, or  $(k+1)$ -double-chain as a trace.*

*Proof.* Suppose  $X$  induces a  $(2, k)$ -star trace with trace sets  $T_1$  and  $T_2$ . Let  $S_i$  be an ancestor in  $\mathcal{H}$  of  $T_i$ . Then

$$|S_i - S_{3-i}| \geq |(S_i - S_{3-i}) \cap X| = |T_i - T_{3-i}| = k$$

for  $i = 1, 2$ , so  $\min\{|S_1 - S_2|, |S_2 - S_1|\} \geq k$ , a contradiction. The statement for  $(2, k)$ -costars follows since a  $(2, k)$ -costar is a  $(2, k)$ -star.

Now suppose  $\mathcal{H}$  contains a  $(k+1)$ -double-chain trace induced by  $X$  with trace set  $T_1, \dots, T_{k+1}$ . By the definition of the double-chain,  $|T_{k+1} - T_1| = |T_1 - T_{k+1}| = k$ , so if  $S_i$  is an ancestor in  $\mathcal{H}$  of  $T_i$ , then as above  $\min\{|S_1 - S_{k+1}|, |S_{k+1} - S_1|\} \geq k$ , a contradiction. Hence  $\mathcal{H}$  has no  $(k+1)$ -double-chain as a trace.  $\blacksquare$

*Proof of Theorem 4.1.1.* Suppose, toward a contradiction, that  $G$  has a  $k$ -MDR; call it  $\mathcal{S} = \{S(v) : v \in V(G)\}$ . Note that  $S(u) = S(v)$  iff  $u = v$ , since with high probability no two distinct vertices of  $G$  have the same neighborhood (by Fact 4.3.2).

By Fact 4.3.1, with high probability,  $G$  contains an independent set  $I = \{v_1, \dots, v_{|I|}\}$  with  $|I| = \lceil \sqrt{\log(n)} \rceil$ . Let  $\mathcal{S}_I = \{S(v_i) : v_i \in I\}$ .

Suppose  $\mathcal{S}_I$  contains a chain of length 3. Without loss of generality  $S(v_1) \subset S(v_2) \subset S(v_3)$ . By Fact 4.3.2 there is a vertex  $x$  such that  $xv_1, xv_3 \in E(G)$  and  $xv_2 \notin E(G)$ . The fact that  $xv_1 \in E(G)$  tells us  $|S(v_1) - S(x)| \geq k$ , so  $|S(v_2) - S(x)| \geq k$  as well. Similarly,  $|S(x) - S(v_2)| \geq |S(x) - S(v_3)| \geq k$ . But then  $xv_2 \in E(G)$ , a contradiction.

Therefore, by Dilworth's Theorem, there is an antichain in  $\mathcal{S}_I$  of size  $\lceil \sqrt{\log n}/2 \rceil$ . Without loss of generality, call this antichain

$$\mathcal{S}^* = \{S(v_i) : 1 \leq i \leq \lceil \sqrt{\log n}/2 \rceil\}.$$

Note that, for all  $S_1, S_2 \in \mathcal{S}^*$ ,  $\min\{|S_1 - S_2|, |S_2 - S_1|\} < k$ .

Take  $n$  large enough that  $\sqrt{\log n}/2 \geq h_{2k}(2, k+1, r)$  with  $r = 20k^4(2k+1)$ . By Theorem 4.2.2,  $\mathcal{S}^*$  contains a  $(2, k)$ -star, a  $(2, k)$ -costar, a  $(k+1)$ -double-chain, or an  $(r, p, q)$ -system with  $p, q \leq k$ ,  $L_1$  empty, and the  $L_i$ 's with  $i \geq 2$  all empty or all nonempty. By Lemma 4.3.3, only the fourth case is possible. Call the sets of this  $(r, p, q)$ -system  $S(v_1), \dots, S(v_r)$ . Let  $V_t = \{v_{20k^4t+1}, \dots, v_{20k^4(t+1)}\}$  for  $0 \leq t \leq 2k$ . Let  $K, F_i, G_i$ , and  $L_i$  be as in the definition of  $(r, p, q)$ -system, so that, for  $1 \leq i \leq r$ ,

$$S(v_i) = K \cup F_i \cup \bigcup_{j \neq i} G_j \cup \bigcup_{j \leq i} L_j.$$

With high probability, we can find a set of vertices  $X = \{x_1, \dots, x_{4k^2}\} \subseteq V(G)$  (applying Fact 4.3.2 for each  $x_i$ ) such that  $x_i$  is adjacent to  $v_j$  if and only if, for some  $0 \leq t \leq 2k$ ,

$$5k^2(i-1) + 1 + 20k^4t \leq j \leq 5k^2i + 20k^4t$$

(from here on we only use the  $v_j$ 's with  $j \leq r$ ) and  $X$  is an independent set in  $G$ . Observe the following two properties:

- (1) for each  $i$  and  $t$ ,  $|N_G(x_i) \cap V_t| = 5k^2$ ,
- (2) for each  $i, j$ , and  $t$  with  $i \neq j$ ,  $|N_G(x_i) \cap N_G(x_j) \cap V_t| = 0$ .

We consider the two cases depending on whether all the  $L_j$ 's with  $j \geq 2$  are empty or nonempty.

**Case 1:**  $L_j = \emptyset$  for all  $1 \leq j \leq r$ . For the proof of this case, we work only with  $v_j$ 's in  $V_0$ .

Fix some  $x_i \in X$ . Define  $f_{i,j} = |F_j \cap S(x_i)|$  and  $g_{i,j} = |G_j - S(x_i)|$ , and let  $f_i = \sum_j f_{i,j}$  and  $g_i = \sum_j g_{i,j}$ . Let  $e_i = |K - S(x_i)|$  and  $c_i = |S(x_i) - \bigcup_j S(v_j)|$ . Then, for any  $v_j \in V_0$ ,

$$|S(v_j) - S(x_i)| = p - f_{i,j} + e_i + g_i - g_{i,j} \tag{4.1}$$

and

$$|S(x_i) - S(v_j)| = f_i - f_{i,j} + c_i + q - g_{i,j}. \quad (4.2)$$

Consider a  $v_j$  not adjacent to  $x_i$ . By the definition of  $k$ -MDR, either

(i)  $|S(v_j) - S(x_i)| < k$  or

(ii)  $|S(x_i) - S(v_j)| < k$ .

We claim there are not both a  $v_{j_1}$  satisfying (i) and a  $v_{j_2}$  satisfying (ii). Indeed, since (i) holds for  $v_{j_1}$ ,  $g_i \leq |S(v_{j_1}) - S(x_i)| + g_{i,j_1} < 2k$ , and, similarly, since (ii) holds for  $v_{j_2}$ , we have  $f_i \leq |S(x_i) - S(v_{j_2})| + f_{i,j_2} < 2k$ . Hence, for a fixed  $i$ , fewer than  $2k$   $f_{i,j}$ 's and fewer than  $2k$   $g_{i,j}$ 's are nonzero (this includes  $v_j$ 's adjacent to  $x_i$ ). Then we can find vertices  $v_{j_3}, v_{j_4}$  with  $f_{i,j_3} = g_{i,j_3} = f_{i,j_4} = g_{i,j_4} = 0$  such that  $x_i$  is adjacent to  $v_{j_3}$  but not to  $v_{j_4}$  (here we use Property 1 from above). But then from Equations 4.1 and 4.2 we have

$$|S(v_{j_4}) - S(x_i)| = |S(v_{j_3}) - S(x_i)| \geq k$$

$$|S(x_i) - S(v_{j_4})| = |S(x_i) - S(v_{j_3})| \geq k,$$

contradicting that  $xv_{j_4} \notin E(G)$ . Therefore, for each  $x_i$ , either

Alternative 1: each  $v_j$  not adjacent to  $x_i$  satisfies (i) and not (ii), or

Alternative 2: each  $v_j$  not adjacent to  $x_i$  satisfies (ii) and not (i).

Now, by the pigeonhole principle, there are either at least  $2k$   $x_i$ 's such that Alternative 1 holds or at least  $2k$   $x_i$ 's such that Alternative 2 holds (we could in fact have  $2k^2$ , but  $2k$  suffices). Assume the former, and without loss of generality let these vertices be  $X_0 = \{x_1, \dots, x_{2k}\}$ . Let  $V' \subseteq V_0$  be the vertices  $v_j$  for which  $G_j$  is contained in  $S(x_i)$  for all  $x_i \in X_0$ . Since (i) holds for all  $x_i \in X_0$ , at most  $2k$   $v_j$ 's satisfy  $G_j \not\subseteq S(x_i)$  for each  $x_i$ , so

$$|V_0 - V'| \leq 2k|X_0| = 4k^2.$$

Therefore  $|N_G(x_i) \cap V'| \geq k^2$  for all  $x_i \in X_0$ .

Fix some  $x_i \in X_0$ . Since Alternative 1 holds for  $x_i$ , we see  $x_i v_j \notin E(G)$  if and only if  $|S(v_j) - S(x_i)| < k$ . For all  $v_j \in V'$ ,  $|S(v_j) - S(x_i)| = p - f_{i,j} + e_i + g_i$  (since  $g_{i,j} = 0$ , by definition of  $V'$ ), and among the terms on the right, only  $f_{i,j}$  depends on  $j$ . Therefore there is a threshold  $t(x_i)$  such that for all  $v_j \in V'$ ,  $x_i$  is adjacent to  $v_j$  if and only if  $f_{i,j} \leq t(x_i)$ . Since each  $x_i \in X_0$  is nonadjacent to some  $v_j \in V'$ , we deduce that  $t(x_i) < p \leq k$ .

Since  $|X_0| > k$ , there are two elements of  $X_0$ , without loss of generality  $x_1$  and  $x_2$ , with  $t(x_1) = t(x_2)$ . Now, for all  $j$  such that  $v_j \in V' \cap N_G(x_1)$  (and hence  $v_j \notin N_G(x_2)$ ),

$$|F_j \cap S(x_1)| \leq t(x_1) = t(x_2) < |F_j \cap S(x_2)|,$$

so there is an element of  $F_j$  in  $S(x_2)$  and not in  $S(x_1)$ . Therefore,

$$|S(x_2) - S(x_1)| \geq |V' \cap N_G(x_1)| \geq k^2.$$

Similarly,  $|S(x_1) - S(x_2)| \geq |V' \cap N_G(x_2)| \geq k^2$ . Hence  $x_1 x_2 \in E(G)$ , a contradiction.

The case when at least  $2k$   $x_i$ 's satisfy Alternative 2 is similar. Call the set of these vertices  $X_0$ . We find a  $V' \subseteq V_0$  such that for all  $x_i \in X_0$  and all  $v_j \in V'$ ,  $S(x_i) \cap F_j = \emptyset$  (that is,  $f_{i,j} = 0$ ) and  $|V' \cap N_G(x_i)| \geq k^2$ . Then for each  $x_i$ , there is a threshold  $t(x_i)$  such that  $x_i$  is adjacent to  $v_j \in V'$  iff  $g_{i,j} \leq t(x_i)$ . We then find two vertices  $x_1$  and  $x_2$  with the same threshold, and necessarily

$$\min\{|S(x_1) - S(x_2)|, |S(x_2) - S(x_1)|\} \geq k^2,$$

so  $x_1$  and  $x_2$  are adjacent in  $G$ , a contradiction. This completes Case 1.

**Case 2:**  $L_1 = \emptyset$  and  $L_j \neq \emptyset$  for all  $j \geq 2$ . Define

$$M_t = \bigcup_{v_j \in V_t} L_j$$

for  $0 \leq t \leq 2k$ . Fix  $x_i \in X$ . If  $S(x_i)$  contains an element of  $M_t$  for each  $t \geq k+1$  and  $M_t - S(x_i)$  is nonempty for each  $t \leq k-1$ , then  $|S(x_i) - S(v_j)|$  and  $|S(v_j) - S(x_i)|$  are at least  $k$  for all  $v_j \in V_k$ , contradicting that  $x_i$  has a nonneighbor in  $V_k$ . Therefore, for each  $x_i$ , one of these cannot happen.

Suppose first that for at least  $2k^2$   $x_i$ 's, it is not the case that  $S(x_i)$  contains an element of  $M_t$  for each  $t \geq k+1$ , so there is a  $t_i \geq k+1$  such that  $S(x_i)$  contains no element of  $M_{t_i}$ . Then there is a set  $X_0$  of  $2k$   $x_i$ 's each with the same  $t_i$ , say  $t_i = m$  for these  $x_i$ 's. Then, if we define  $b_i = |\bigcup_{t < m} M_t - S(x_i)|$  and follow the notation of Equations 4.1 and 4.2, we have for all  $v_j \in V_m$ ,

$$|S(v_j) - S(x_i)| = p - f_{i,j} + e_i + g_i - g_{i,j} + b_i + \left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha \leq j}} L_\alpha - S(x_i) \right|.$$

But

$$\left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha \leq j}} L_\alpha - S(x_i) \right| = \left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha \leq j}} L_\alpha \right| \geq j - 20k^4m$$

since  $L_j$  is nonempty for  $v_j \in V_m$  and  $S(x_i) \cap M_m = \emptyset$ . Therefore  $|S(v_j) - S(x_i)| \geq k$  for all but the  $k$  smallest-indexed  $v_j$ 's in  $V_m$ ; let  $V^*$  be the set of  $v_j \in V_m$  with  $|S(v_j) - S(x_i)| \geq k$ . Observe that, for all  $v_j \in V^*$  and  $x_i \in X_0$ ,

$$\begin{aligned} |S(x_i) - S(v_j)| &= f_i - f_{i,j} + c_i + q - g_{i,j} + s_i + \left| \bigcup_{\substack{v_\alpha \in V_m \\ \alpha > j}} L_\alpha \cap S(x_i) \right| \\ &= f_i - f_{i,j} + c_i + q - g_{i,j} + s_i, \end{aligned}$$

where  $s_i = |\bigcup_{t > m} M_t \cap S(x_i)|$ . Fix  $x_i \in X_0$  and let  $v_j \in V^*$  be a nonneighbor of  $x_i$ , so necessarily  $|S(x_i) - S(v_j)| < k$ . Since  $g_{i,j} \leq q$ , it follows that,  $f_i \leq |S(x_i) - S(v_j)| + f_{i,j} \leq 2k - 1$ . Therefore, for each  $i$ , at most  $2k - 1$   $f_{i,j}$ 's are nonzero. Let  $V' \subseteq V^*$  be the vertices  $v_j$  for which every  $x_i \in X_0$  has  $f_{i,j} = 0$ . Then

$$|V^* - V'| \leq (2k - 1)|X_0| \leq 4k^2 - 2k,$$

so each  $x_i \in X_0$  has at least

$$5k^2 - |V_m - V^*| - (4k^2 - 2k) \geq k^2$$

neighbors in  $V'$ . Since  $f_{i,j} = 0$  for all  $x_i \in X_0$  and  $v_j \in V'$ , we see that, for a fixed  $i$ ,  $|S(x_i) - S(v_j)|$  depends only on  $g_{i,j}$ . Thus there is a threshold  $t(x_i) \leq k$  such that  $v_j$  is adjacent to  $x_i$  if and only if  $g_{i,j} \leq t(x_i)$ . Since  $|X_0| = 2k$ , there are two elements of  $X_0$ , say  $x_1$  and  $x_2$ , with the same threshold. As in the proof of Case 1, it can be proved that  $x_1$  and  $x_2$  are adjacent, a contradiction.

Finally, we are left with the subcase that there are at least  $2k^2$   $x_i$ 's such that  $M_t - S(x_i)$  is empty for some  $t \leq k - 1$ . This is similar to the previous subcase, except that the roles of  $|S(x_i) - S(v_j)|$  and  $|S(v_j) - S(x_i)|$  are reversed; we skip the details.  $\square$

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- *Undergrad Math Conference*, Rose-Hulman Institute of Technology, Terra Haute, IN, *March 2003*
- *16th Cumberland Combinatorics Conference*, Georgia Institute of Technology, Atlanta, GA, *May 2003*
- *Combinatorics Seminar*, Univ. of Illinois, Urbana, IL, *September 2003*
- *Undergrad Math Conference*, Rose-Hulman Institute of Technology, Terra Haute, IN, *March 2004*
- *Combinatorics Seminar*, Eidgenössische Technische Hochschule (ETH), Zürich, Switzerland, *May 2004*
- *Combinatorics Seminar*, Univ. of Memphis, Memphis, TN, *December 2004*

- *Combinatorics Seminar*, Univ. of Illinois, Urbana, IL, *March 2005*
- *Combinatorics Seminar*, Univ. of Memphis, Memphis, TN, *March 2005*
- *Discrete Math Seminar*, Illinois State Univ., Normal, IL, *February 2006*
- *Combinatorics Seminar*, Univ. of Illinois, Urbana, IL, *March 2006*
- *19th Cumberland Combinatorics Conference*, East Tennessee State Univ., Johnson City, TN, *May 2006*
- *20th Cumberland Combinatorics Conference*, Emory University, Atlanta, GA, *May 2007*
- *Combinatorics Seminar*, Univ. of Illinois, Urbana, IL, *October 2007*
- *AMS Sectional Meeting # 1038 (invited speaker)*, Indiana University, Bloomington, IN, *April 2008*

#### TEACHING EXPERIENCE

Math 231: Calculus 2	Teaching Assistant	Fall 2005
Math 242: Calculus 3	Teaching Assistant	Spring 2006
Math 231: Calculus 2	Teaching Assistant	Fall 2006
Math 231: Calculus 2	Instructor	Summer 2007
Math 231: Calculus 2	Merit Teaching Assistant	Fall 2007

#### EMPLOYMENT

- **Fellow**, Dept. of Mathematics, Univ. of Illinois, *Spring 2008*
- **Teaching Assistant**, Univ. of Illinois, Urbana, IL, *Fall 2005-Fall 2007*
- **Research Assistant**, Univ. of Illinois, Urbana, IL, *Spring 2007*
- **Van Vleet Fellow**, Univ. of Memphis, Memphis, TN, *Fall 2004-Spring 2005*

#### HONORS

- **PARKER AND SCHARK FELLOW**, Department of Mathematics, Univ. of Illinois, *Spring 2008*

- RESEARCH ASSISTANT FOR CAMPUS RESEARCH BOARD, Univ. of Illinois, *Spring 2007*
- NAMED TO INCOMPLETE LIST OF TEACHERS RANKED EXCELLENT BY THEIR STUDENTS, Univ. of Illinois, *Fall 2006*
- NAMED TO INCOMPLETE LIST OF TEACHERS RANKED EXCELLENT BY THEIR STUDENTS WITH OUTSTANDING RATINGS, Univ. of Illinois,, *Spring 2006*
- VAN VLEET FELLOW, Univ. of Memphis, *Fall 2004-Spring 2005*
- H.R. BRAHANA PRIZE FOR OUTSTANDING GRADUATING SENIOR, Dept. of Mathematics, Univ. of Illinois, *Spring 2004*
- GRADUATED WITH HIGHEST DISTINCTION, Dept. of Mathematics, Univ. of Illinois, *Spring 2004*
- DEAN'S LIST, JAMES SCHOLAR, CUM LAUDE, Univ. of Illinois,

#### PROFESSIONAL ACTIVITIES

Referee for journals *Combinatorics, Probability, and Computing; Order; Discrete Mathematics*