

Nordhaus-Gaddum Bounds for k -Domination in Graphs

Noah Prince^{*†}

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Abstract

A k -dominating set of a graph G is a set S of vertices of G such that every vertex outside of S has k neighbors in S . The k -domination number of G , written $\gamma_k(G)$, is the size of the smallest k -dominating set in G . In this paper, we derive sharp upper and lower bounds on $\gamma_k(G) + \gamma_k(\overline{G})$ and $\gamma_k(G)\gamma_k(\overline{G})$, where \overline{G} is the complement of G . We use the results for $k = 2$ to prove a conjecture of Alon, Balogh, Bollobás, and Szabó on game domination numbers.

1 Introduction

Domination is a fundamental concept in graph theory which has been studied extensively; for history and surveys of the problem see [8, 9]. In this paper, we study a generalization of the domination problem.

For a graph G , a set $S \subseteq V(G)$ is a k -dominating set of G if every vertex in $V(G) - S$ has k neighbors in S . The k -domination number of G , written $\gamma_k(G)$, is the minimum size of a k -dominating set of G . Under this definition, the usual domination number $\gamma(G)$ is $\gamma_1(G)$.

For bounds on γ_k , trivially we have $k \leq \gamma_k(G) \leq n$ for all graphs G of order $n \geq k$. The upper bound is achieved whenever the maximum degree of G is less than k . If we rule out small degree vertices, we can improve upon this bound.

^{*}Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL, 61801

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Theorem 1.1 ([2, 5]) *If G is a graph of order n with minimum degree at least k , then $\gamma_k(G) \leq kn/(k+1)$.*

For a graph parameter P , Nordhaus-Gaddum bounds refer to bounds on $P(G) + P(\overline{G})$ or $P(G)P(\overline{G})$, (where \overline{G} is the complement of G), usually in terms of the order $n(G)$. Results of this type have a Ramsey flavor, and have been proved for many parameters. In the case $k = 1$, these bounds on γ_k were discovered by Jaeger and Payan.

Theorem 1.2 ([10]) *For any graph G on n vertices,*

$$\gamma(G) + \gamma(\overline{G}) \leq n + 1$$

$$\gamma(G)\gamma(\overline{G}) \leq n.$$

The goal of this paper is to generalize Jaeger and Payan's result to $k > 1$.

Main Result. *For $k \geq 2$ an integer and G a graph of order $n \geq n_0(k)$,*

$$8k/3 \leq \gamma_k(G) + \gamma_k(\overline{G}) \leq n + 2k - 1$$

$$16k^2/9 \leq \gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2).$$

Furthermore, these bounds are sharp.

The Main Result will be established in several pieces, namely, Theorems 2.1, 6.1, and 6.5.

The function $n_0(k)$ appearing in the Main Result is small for three of the inequalities. Indeed, the lower bounds need only $n \geq 3k$, and the upper sum bound is valid for all n . The upper bound on products, however, requires n to be $\exp(O(k \log k))$. For smaller values of n , we can prove a weaker product upper bound of the same order of magnitude (that is, $O(kn)$).

When the ratio $\Delta(G)/\delta(G)$ is small enough (for example, for regular graphs) and n is large, we can improve the product upper bound to

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (k+1)n,$$

and if in addition $\delta(G) \geq k$,

$$\gamma_k(G)\gamma_k(\overline{G}) \leq kn.$$

This constitutes Theorem 7.1.

In the case $k = 2$, we can use some of our results to prove a conjecture of Alon, Balogh, Bollobás, and Szabó concerning game domination.

The proofs we present employ a variety of methods, relying heavily on probability and Δ -systems.

The organization of the paper is as follows. In Section 2, we derive the lower Nordhaus-Gaddum bounds for γ_k . In Sections 3 and 4, we prove a handful of lemmas and propositions which will be needed for the upper bounds. In Section 5, we develop Δ -systems as a tool for Section 6, we derive both the upper Nordhaus-Gaddum bounds for γ_k mentioned in the Main Result and a weaker product bound valid for all values of n . Section 7 improves the bounds of the section prior when the ratio $\Delta(G)/\delta(G)$ is small. Finally, in Section 8, we introduce the concepts of game domination and game domination number $\gamma_g(G)$, and use some of our earlier results to prove a conjecture about γ_g .

Before we begin, we fix notation. All graphs considered in this paper are finite and simple. For a graph G , $V(G)$ is its vertex set and $n(G) = |V(G)|$. For disjoint $X, Y \subseteq V(G)$, $E_G(X, Y)$ denotes the set of edges with one endpoint in X and the other in Y . For a set $X \subseteq V(G)$, $N_G^k(X)$ is the subset of $V(G) - X$ whose elements have at least k neighbors in X , so the neighborhood in G of a vertex v , $N_G(v)$, is $N_G^1(\{v\})$. For $N_G^k(X) \cup X$ we write $N_G^k[X]$. The set X k -dominates $v \in V(G)$ iff $v \in N_G[X]$. A *set system* is a family of finite sets. For a graph G , \mathcal{N}_G is the set system $\{N_G(v) : v \in V(G)\}$ (the members of \mathcal{N}_G appear with multiplicity). The degree of vertex v in G is $d_G(v)$. For the maximum and minimum degrees of G we use $\Delta(G)$ and $\delta(G)$, respectively. For further background and notation, see [12].

2 Lower Bounds

In this section we present the following Theorem giving Nordhaus-Gaddum-type lower bounds for γ_k .

Theorem 2.1 *Let G be a graph of order $n \geq 3k$. Then*

$$\gamma_k(G) + \gamma_k(\overline{G}) \geq 8k/3$$

and

$$\gamma_k(G)\gamma_k(\overline{G}) \geq 16k^2/9.$$

Proof. Let A be a minimum k -dominating set of G , B be a minimum k -dominating set of \overline{G} , and $C = A \cap B$. Let $a = |A|$, $b = |B|$, and $c = |C|$.

Suppose first that $A \cup B = V(G)$. Then, since γ_k is at least k for G and \overline{G} , we have

$$\gamma_k(G)\gamma_k(\overline{G}) \geq k(n - k) \geq 2k^2,$$

which forces $\gamma_k(G) + \gamma_k(\overline{G}) \geq 2\sqrt{2}k$, so the desired bounds hold. We therefore assume $A \cup B \neq V(G)$.

Now suppose that $A = B = C$. Then, since each vertex outside C sends $2k$ edges into C between G and \overline{G} , we have $|C| \geq 2k$, so

$$\gamma_k(G)\gamma_k(\overline{G}) = |C|^2 \geq 4k^2.$$

From this it follows that $\gamma_k(G) + \gamma_k(\overline{G}) \geq 4k$. We may therefore assume $A \neq B$, so that in particular $a + b - 2c \neq 0$.

Now, in G , each vertex of $B - A$ has at least k neighbors in A , and, similarly, each vertex of $A - B$ has at least k neighbors in B . Therefore, by counting edges, we have

$$(a - c)k + (b - c)k \leq |A - B||B| + |B - A||A| - |A - B||B - A| = ab - c^2. \quad (1)$$

We also know that in G each vertex of $A - C$ has at least $k - c$ neighbors in $B - C$, and in \overline{G} each vertex of $B - C$ has at least $k - c$ neighbors in $A - C$. By counting possible edges between $A - C$ and $B - C$, we have

$$(k - c)(a + b - 2c) \leq (a - c)(b - c) \leq \frac{1}{4}(a + b - 2c)^2.$$

Dividing through by $a + b - 2c$ shows $a + b - 2c \geq 4k - 4c$. Furthermore, because every vertex in neither A nor B sends $2k$ edges into $A \cup B$ between G and \overline{G} , we see $|A \cup B| = a + b - c \geq 2k$. Plugging this information into Equation 1,

we have
$$\begin{aligned} ab &\geq k(4k - 4c) + c^2 \\ ab &\geq k(2k - c) + c^2. \end{aligned}$$
 The maximum of these two quadratics in

c is minimized at $c = 2k/3$, giving us that $ab = \gamma_k(G)\gamma_k(\overline{G}) \geq 16k^2/9$. This immediately implies that $\gamma_k(G) + \gamma_k(\overline{G}) \geq 8k/3$, so we are done. \square

The bounds in Theorem 2.1 are sharp when k is divisible by 3, as shown by the example of Figure 1. Let A and B be sets of $4k/3$ vertices whose intersection C is the disjoint union of two sets C_1 and C_2 , each of size $k/3$. Lines between sets represent edges in G , whereas dotted lines are edges of \overline{G} . Between $A - C$ and $B - C$, place a $k/3$ -regular bipartite graph in G (and, by symmetry, in \overline{G}). All other edges are arbitrary. It is easy to see A and B are k -dominating sets for G and \overline{G} , respectively.

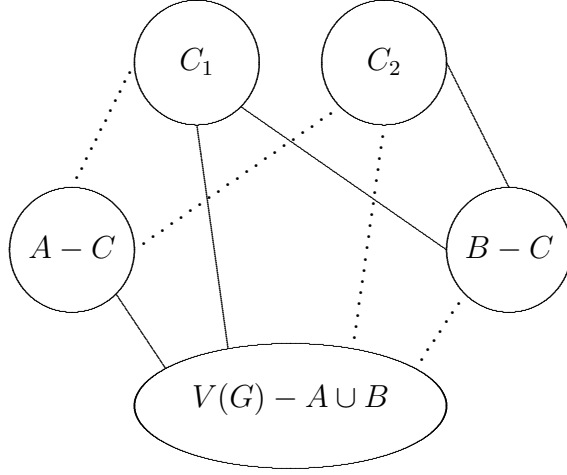


Figure 1

3 Some Lemmas

We now set our sights on proving Nordhaus-Gaddum-type upper bounds. Before we can, though, we must prove several lemmas. The first two employ the probabilistic method. We use \mathbb{P} to denote probability and \mathbb{E} or expectation.

Theorem 3.1 *Let $k \geq 2$ be an integer and G be a graph of order n . Suppose that at most m vertices of G have degree less than $d \geq 7k \log k$. Then*

$$\gamma_k(G) \leq \left(\frac{k \log d}{d} + \frac{k(2e \log d)^{k-1}}{d^k} \right) n + m.$$

Proof. Let M be the set of vertices of degree less than d in G . Choose a set $X \subseteq V(G)$ by putting each vertex of G in X independently with probability $p = k \log d/d$, so $\mathbb{E}(|X|) = pn$. Note that since $d \geq 7k \log k$, $p < 1/2$.

Let Y be the set of vertices of $V(G) - M$ with fewer than k neighbors in X . For each vertex v , let N_v be a set of d neighbors of v . Then

$$\mathbb{P}(v \in Y) \leq \mathbb{P}(|N_v \cap X| < k) = \sum_{i=0}^{k-1} \binom{d}{i} p^i (1-p)^{d-i}$$

$$\leq (1-p)^d \sum_{i=0}^{k-1} \frac{(dp)^i}{i!} \left(\frac{1}{1-p} \right)^{i-1} < \frac{2^{k-1}}{d^k} \sum_{i=0}^{k-1} \frac{(k \log d)^i}{i!}$$

using that $(1-p)^d \leq e^{-pd} = 1/d$ and $1/(1-p) < 2$. The maximum term of the sum is that corresponding to $i = k-1$, so the sum is at most

$$k(k \log d)^{k-1}/(k-1)! \leq k(2e \log d)^{k-1}.$$

Therefore

$$\mathbb{E}(Y) = n\mathbb{P}(v \in Y) \leq n \cdot \frac{k(2e \log d)^{k-1}}{d^k}.$$

Since $X \cup M \cup Y$ is a k -dominating set with expected size $pn + m + \mathbb{E}(Y)$, there is some k -dominating set with at most the desired size. \square

It should be noted that when d is much larger than k , then setting $p = (\log d + (k-1) \log \log d)/d$ gives a better bound. However, for small values of d , the ease of presentation justifies any difference in the bounds.

Lemma 3.2 *Let G be a graph of order n . Suppose there is a set of q vertices which k -dominates all vertices of G with degree less than $40k$. Then $\gamma_k(G) < 21n/100 + q$.*

Proof. Repeat the proof of Lemma 3.1 with $p = 1/5$, so $\mathbb{E}(X) = n/5$. Let Y be the set of vertices with degree at least $40k$ which are not k -dominated by X . Then

$$\mathbb{P}(v \in Y) \leq k \left(\frac{4}{5} \right)^d \frac{(ed)^{k-1}}{(4(k-1))^{k-1}} \leq k \left(\frac{4}{5} \right)^d (10e)^{k-1} < \frac{k}{100}.$$

This shows that $\mathbb{E}(|X| + |Y|) < 21n/100$, and adding q vertices to k -dominate the small-degree vertices shows there is a k -dominating set with at most the desired size. \square

We can, of course, apply the Lemma above with q equal to the number of vertices of degree less than $40k$. It will be advantageous during the proof of Lemmas 3.3 and 6.4, though, to exploit the additional generality we have afforded ourselves.

Lemma 3.3 *Let G be a graph of order $n \geq 6400$ with $\gamma_2(\overline{G}) \leq 7$ and $\gamma_2(G)\gamma_2(\overline{G}) > 2n$. Then $\gamma_2(\overline{G}) \leq 4$.*

Proof. Let $M = \{v : d_G(v) < 80\}$. The hypotheses imply that $\gamma_2(G) > 2n/7$, so by Lemma 3.2 with $q = |M|$, we have $|M| > n/20$. Among pairs of vertices of M , let v_1 and v_2 minimize $|N_G(v_1) \cap N_G(v_2)|$.

If $U = N_G(v_1) \cup N_G(v_2)$ 2-dominates all but 160 vertices of M , then by Lemma 3.2

$$\gamma_2(G) \leq 21n/100 + |U| + 160 \leq 2n/7,$$

a contradiction. Hence at least 161 vertices of M have at most one neighbor in U , and so two of them, say v_3 and v_4 , are adjacent to all of $U - x$ for some $x \in U$. Since $|N_G(v_1) \cap N_G(v_3)| \leq 1$ we see by minimality that v_1 and v_2 have at most one common neighbor.

Now, in \overline{G} , v_1 and v_3 2-dominate $V(G) - N_G(v_1) - N_G(v_3)$ and v_2 and v_4 2-dominate $N_G(v_1) - x$ for $i = 1, 3$. This shows that $\{v_1, v_3, v_2, v_4, x\}$ is a 2-dominating set of size 4 in \overline{G} , which completes the proof. \square

Our fourth lemma is an easy corollary of the following theorem of Erdős, Faudree, Gyárfás, and Schelp.

Theorem 3.4 ([6]) *For any graph G on n vertices, either $\gamma(G)$ or $\gamma(\overline{G})$ is at most $\lceil \log n \rceil$.*

Lemma 3.5 *For any graph G on n vertices and any integer $k \geq 1$, either $\gamma_k(G)$ or $\gamma_k(\overline{G})$ is at most $(2k - 1)\lceil \log n \rceil$.*

Proof. Inductively define X_i to be a dominating set of $G - \bigcup_{j=1}^{i-1} X_j$ or $\overline{G} - \bigcup_{j=1}^{i-1} X_j$ of size at most $\lceil \log n \rceil$, which exists by Theorem 3.4. Let $X = \bigcup_{j=1}^{2k-1} X_j$.

By the Pigeonhole Principle, at least k X_i 's dominate $G - X$ or at least k X_i 's dominate $\overline{G} - X$, so either $\gamma_k(G)$ or $\gamma_k(\overline{G})$ is at most $|X| = (2k - 1)\lceil \log n \rceil$. \square

The final lemma of this section comes essentially from [5], where it was proven for $m = 0$. The proof is practically the same; we omit the details.

Lemma 3.6 ([5]) *Let w and k be positive integers. If G is a graph of order n with at most m vertices of degree less than $(w + 1)k/w - 1$, then $\gamma_k(G) \leq wn/(w + 1) + m$.*

4 Some Computations

In this Section we collect two propositions whose proofs involve simple computation. Doing this work here will facilitate the proof of Lemma 6.4.

Proposition 4.1 *Let r and $k \geq 2$ be positive integers. Let $w = \lfloor r/(r - k + 1) \rfloor$. Then $\lfloor wk/(w + 1) - 1 \rfloor < r$.*

Proof. By definition $\lfloor wk/(w + 1) - 1 \rfloor = k - 1 + \lfloor k/w \rfloor$, so we are done as long as $\lfloor k/w \rfloor < r - k + 1$. This is true if $k/w < r - k + 1$, which is equivalent to

$$k/(r - k + 1) < \lfloor r/(r - k + 1) \rfloor = 1 + \lfloor (k + 1)/(r - k + 1) \rfloor.$$

This holds because $\lfloor x \rfloor > x - 1$ for all real x . □

Proposition 4.2 *Let r and $k \geq 2$ be positive integers satisfying $k \leq r \leq 2k - 1$. Let $w = \lfloor r/(r - k + 1) \rfloor$ and let $m = (k + r)w/(w + 1)$. Then $m \leq 2k - 4/3$.*

Proof. First we consider the cases when $w \leq 3$. If $w = 1$, then $r = 2k - 1$, so $m = (3k - 1)/2 < 2k - 4/3$. If $w = 2$, then $r \leq 2k - 2$ so $m \leq 2(3k - 2)/3 = 2k - 4/3$. If $w = 3$, then $r \leq 3(k - 1)/2$, so $m \leq 15k/8 - 9/8 < 2k - 4/3$. In all three cases, the desired bound holds.

Now consider $w \geq 4$, so that $r \leq 4(k - 1)/3$. Then we have

$$\begin{aligned} m &= (k + r) \left(1 - \frac{1}{w + 1} \right) = (k + r) \left(1 - \frac{1}{\lfloor (2r - k + 1)/(r - k + 1) \rfloor} \right) \\ &\leq (k + r) \frac{r}{2r - k + 1}, \end{aligned}$$

where the inequality comes from ignoring the floor signs in the denominator. The derivative of the right hand side above is negative for $r \leq 4(k - 1)/3$, so m is maximized when $r = k$. At $r = k$ we have $w = k$ and $m = 2k^2/(k + 1) \leq 2k - 4/3$, which proves the proposition. □

5 Δ -systems

A Δ -system is a set system $\mathcal{A} = \{A_1, \dots, A_\ell\}$ such that there is a set K with $A_i \cap A_j = K$ whenever $i \neq j$. K is called the *kernel* of \mathcal{A} . An (r, ℓ) - Δ -system is one which additionally satisfies $|\mathcal{A}| = \ell$ and $|A| = r$ for all $A \in \mathcal{A}$. For a survey of results on Δ -systems, see [11].

The most famous question on (r, ℓ) - Δ -systems is how large an r -uniform set system must be in order to contain one. Erdős and Rado gave the following bound for this problem.

Theorem 5.1 ([7]) *Any r -uniform set system with at least $(r-1)^\ell \ell!$ members contains an (r, ℓ) - Δ -system.*

We shall apply Theorem 5.1 to find (r, ℓ) - Δ -systems in the set system \mathcal{N}_G as defined in the Introduction. To this end, we make the following observation.

Proposition 5.2 *Let r , k , and q be positive integers and let G be a graph with at least $2(kq)^q$ vertices of degree at most q . Then \mathcal{N}_G contains an $(r, k+1)$ - Δ -system for some $r \leq q$.*

Proof. By our assumptions, the family $\mathcal{N}_G^* = \{N_G(v) : d_G(v) \leq q\}$ (with multiplicities) satisfies

$$|\mathcal{N}_G^*| \geq 2(kq)^q > \sum_{r=0}^q k^r r!,$$

so by the pigeonhole principle and Theorem 5.1, \mathcal{N}_G must contain an $(r, k+1)$ - Δ -system for some $r \leq q$. \square

The presence of certain Δ -systems in \mathcal{N}_G forces $\gamma_k(\overline{G})$ to be small.

Proposition 5.3 *Suppose G is a graph and \mathcal{N}_G contains an $(r, k+1)$ - Δ -system with kernel K . If $|K| = r$, then $\gamma_k(\overline{G}) \leq |K| + k$. If $|K| < r$, then $\gamma_k(G) \leq |K| + k + 1$.*

Proof. Let $\mathcal{A} = \{N_G(v_1), \dots, N_G(v_{k+1})\}$ be an $(r, k+1)$ - Δ -system in \mathcal{N}_G , and let K be the kernel of \mathcal{A} . First observe that v_1, \dots, v_k k -dominate $V(G) - \bigcup_{i=1}^{k+1} N_G(v_i)$ in \overline{G} . If $|K| < r$, take the set $K \cup \{v_1, \dots, v_{k+1}\}$; it is a k -dominating set of \overline{G} from our observation above and because v_i dominates $N_G(v_j) - K$ in \overline{G} if $i \neq j$. If $|K| = r$, then $N_G(v_i) - K = \emptyset$ for $1 \leq i \leq k+1$, so $K \cup \{v_1, \dots, v_k\}$ is a k -dominating set of \overline{G} . \square

6 Upper Bounds

In this section we prove upper bounds on $\gamma_k(G) + \gamma_k(\overline{G})$ and $\gamma_k(G)\gamma_k(\overline{G})$. We begin by proving two theorems that apply for all values of $n(G)$. After that, we prove two more lemmas before establishing Theorem 6.5, the culmination of Sections 3 through 6.

Theorem 6.1 *For any graph G on n vertices, $\gamma_k(G) + \gamma_k(\overline{G}) \leq n + 2k - 1$.*

Proof. Let $X \subseteq V(G)$ have size $2k - 1$. By the pigeonhole principle, $N_G^k(X)$ and $N_{\overline{G}}^k(X)$ are disjoint and have union $V(G) - X$. Therefore, $N_G^k[X]$ k -dominates G and $N_{\overline{G}}^k[X]$ k -dominates \overline{G} , so

$$\gamma_k(G) + \gamma_k(\overline{G}) \leq |N_G^k[X]| + |N_{\overline{G}}^k[X]| = n + |X| = n + 2k - 1.$$

□

Theorem 6.1 is sharp, at least for small n . For example, when k is odd, let G be a $(k - 1)$ -regular graph on $2k - 1$ vertices. Then $\gamma_k(G) + \gamma_k(\overline{G}) = 2n = n + 2k - 1$.

Theorem 6.2 *Let $k \geq 2$ be an integer and G be a graph of order n . Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq 8kn.$$

Proof. Note first that if $\gamma_k(G) \leq 8k$ we are done, so assume otherwise. Partition $V(G)$ into $m = \lceil 4n/3\gamma_k(G) \rceil$ sets V_1, \dots, V_m of size at most $3\gamma_k(G)/4$. Note that $m \leq 2n/\gamma_k(G)$.

No V_i k -dominates at least $n - 2k + 3$ vertices in G , otherwise

$$\gamma_k(G) \leq |V_i| + 2k - 3 < 3\gamma_k(G)/4 + \gamma_k(G)/4 = \gamma_k(G).$$

Therefore there is a set D_i of at least $2k - 2$ vertices which send at most $k - 1$ edges in G into V_i for each $1 \leq i \leq m$. Let $U_i \subseteq V_i$ be the vertices not k -dominated in \overline{G} by D_i . Then $|U_i| \leq |E_G(V_i, D_i)|/(k - 1) \leq |D_i|$.

Now $D_i \cup U_i$ k -dominates V_i in \overline{G} , so

$$\gamma_k(\overline{G}) \leq \left| \bigcup_{i=1}^m (D_i \cup U_i) \right| \leq (4k - 4)m \leq 4k \cdot \frac{2n}{\gamma_k(G)}.$$

Therefore $\gamma_k(G)\gamma_k(\overline{G}) \leq 8kn$. \square

It is unclear whether the constant 8 in Theorem 6.2 can be improved. The complete bipartite graph $K_{k-1, n-k+1}$ satisfies $\gamma_k(G) = n - k + 1$ and $\gamma_k(\overline{G}) = 2k - 1$ if $n \geq 2k - 1$. This tells us the 8 should be at least about 2. $K_{k-1, n-k+1}$ is nearly optimal, but not quite as good as the following construction.

Construction. Let $G \wedge H$ denote the G join H , the graph formed by taking disjoint copies of G and H and adding all possible edges between the two. Let mG be the disjoint union of m copies of G . Define $G_{n,k} = K_{k-2} \wedge \frac{n-k+2}{2}K_2$.

We claim $\gamma_k(G_{n,k})\gamma_k(\overline{G_{n,k}}) = (2k - 1)(n - k + 2)$ if k is odd. Since $G_{n,k}$ has only $k - 2$ vertices of degree at least k , $\gamma_k(G_{n,k}) \geq n - k + 2$, and equality clearly holds. In \overline{G} , the $k - 2$ vertices of the K_{k-2} in G must be in a k -dominating set since they have degree less than k . We need to include at least $k + 1$ vertices from the $\frac{n-k+2}{2}K_2$, since any k of them fail to k -dominate at least one vertex (that one vertex being one appearing in a K_2 in G with one of the k chosen vertices). Therefore $\gamma_k(\overline{G}) \geq (2k - 1)$, and it is easy to see equality holds here as well. This establishes the claim. \square

We separate the bulk of the argument for Theorem 6.5 into the next two lemmas.

Lemma 6.3 *Let G be a graph of order $n \geq 10^6k^2$, and suppose that any $(2k - 1)$ -subset X of $V(G)$ satisfies that $N_G^k(X)$ k -dominates all but at most $2k - 2$ vertices of $V(G) - N_G^k[X]$. Then $\gamma_k(G)\gamma_k(\overline{G}) \leq kn$.*

Proof. Let X be the set of $2k - 1$ vertices of G with smallest degree sum. Let $d = |N_G^k(X)|$ and $R = V(G) - N_G^k[X] = N_G^k(X)$.

Fix a partition $\{A_1, \dots, A_q, A^*\}$ of $N_G^k(X)$ such that A_i is a minimal set which k -dominates all but at most $2k - 2$ vertices of R for $1 \leq i \leq q$ and at least $2k - 1$ vertices of R are not k -dominated by A^* . Note that since $N_G^k(X)$ k -dominates all but at most $2k - 2$ vertices of R by hypothesis, $q \geq 1$.

Now, X , the smallest A_i , and the vertices of R not k -dominated by that smallest A_i are a k -dominating set of G , so

$$\gamma_k(G) \leq |X| + \min_{1 \leq i \leq q} |A_i| + (2k - 2) \leq 4k - 3 + \frac{d}{q}.$$

In \overline{G} , X k -dominates R . Also, for any given A_i and any $v_i \in A_i$, we know by minimality of A_i that there is a subset S_i of R of $2k - 2$ vertices which

send at most $k - 1$ nonedges in \overline{G} into $A_i - v_i$. If T_i is the set of vertices of $A_i - v_i$ which are not k -dominated in \overline{G} by S_i , then

$$|T_i| \leq \frac{|E_G(S_i, A_i)|}{k-1} \leq \frac{|S_i|(k-1)}{k-1} \leq 2k-2.$$

Note that $S_i \cup T_i \cup \{v_i\}$ is a k -dominating set of A_i in \overline{G} with size at most $4k - 3$. Finally, as with the A_i 's, there are sets $S^* \subseteq R$ and $T^* \subseteq A^*$, each of size at most $2k - 2$, such that $S^* \cup T^*$ k -dominate A^* in \overline{G} . Therefore

$$\gamma_k(\overline{G}) \leq |X| + \sum_{i=1}^q (|S_i \cup T_i| + 1) + |S^* \cup T^*| \leq (4k - 3)q + 6k - 5.$$

Putting these together, we have

$$\gamma_k(G)\gamma_k(\overline{G}) \leq 4kd + 16k^2q + \frac{6kd}{q} + 24k^2 \leq 20kd + 30k^2$$

since $q \leq d/k$ and $k \leq d$. Then $\gamma_k(G)\gamma_k(\overline{G}) \leq kn$ unless $d \geq n/30$.

If $d \geq n/30$, the sum of the degrees of vertices in X is at least kd , so at most $2k - 2$ vertices of X have degree at most $kd/(2k - 1) \geq n/60$ in G . But since the argument above holds for all $(2k - 1)$ -sets X , at most $2k - 2$ vertices of G can have degree at most $kd/(2k - 1)$. By Theorem 3.1, since $kd/(2k - 1) \geq n/60$, $\gamma_k(G) \leq 60k \log n$. Since the arguments above are symmetric in G and \overline{G} , the same bound holds for $\gamma_k(\overline{G})$. But then

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (60k \log n)^2 \leq kn,$$

so we are done. \square

Before continuing to our next lemma, we describe how Lemma 6.3 can be used to derive Nordhaus-Gaddum bounds for other parameters which are domination-like. The method of Lemma 6.3 first appeared for γ in [3], where it was shown that if $\gamma(G)\gamma(\overline{G})$ is not small, then G or \overline{G} has diameter at least 3. ‘‘Small’’ in that sense meant $O((n \log n)^{2/3})$. If P is some parameter satisfying $c_1\gamma(G) \leq P(G) \leq c_2\gamma(G)$ for some constants c_1 and c_2 , then $\gamma(G)\gamma(\overline{G}) = O((n \log n)^{2/3})$ as well unless G or \overline{G} has diameter at least 3. On the other hand,

$$P(K_n)P(\overline{K_n}) \geq c_1^2\gamma(K_n)\gamma(\overline{K_n}) = c_1^2n = \omega((n \log n)^{2/3}),$$

so if G is extremal for $P(G)P(\overline{G})$, G or \overline{G} must have diameter at least 3. To derive the Nordhaus-Gaddum bounds on P , it suffices to consider only such graphs. This method was used for the parameters γ and the Roman domination number γ_R in [3] and for the mobile eternal security number σ_m in [4] (we have $\gamma(G) \leq \sigma_m(G) \leq \gamma_R(G) \leq 2\gamma(G)$ for all G).

By being more careful in the proof of Lemma 6.3, we can show that $\gamma_k(G)\gamma_k(\overline{G}) = O((kn \log n)^{2/3})$ if the conditions of Lemma 6.3 apply. Therefore, as above, if P is some parameter bounded between constants times γ_k , $P(G)P(\overline{G})$ is maximized when the conditions of Lemma 6.3 fail for G or \overline{G} . To find Nordhaus-Gaddum bounds for P , we need only consider these graphs; this is what we do in Section 8 for the game domination number γ_g , which satisfies $\gamma(G) \leq \gamma_g(G) \leq \gamma_2(G)$.

Lemma 6.4 *Suppose $k \geq 2$ is an integer and G is a graph of order $n \geq 72(40k)^{80k}$. If $\gamma_k(\overline{G}) \leq 6k - 5$, then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2). \quad (2)$$

Proof. Suppose that $\gamma_k(G)\gamma_k(\overline{G}) \geq (2k - 1)(n - k + 1)$. By assumption on $\gamma_k(\overline{G})$, we have $\gamma_k(G) \geq (n - k + 1)/3$.

Let $M = \{v : d_G(v) < 40k\}$. By Lemma 3.2 with $q = |M|$, we have $|M| \geq n/20$. Then, by Proposition 5.2 the set system \mathcal{N}_G contains an $(r, k + 1)$ - Δ -system $D = \{N(v_i) : 1 \leq i \leq k + 1\}$ for some $r < 40k$. Let K be the kernel of \mathcal{D} . As in the proof of Proposition 5.3, the vertices v_1, \dots, v_{k+1} k -dominate $V(G) - K$ in \overline{G} . If K k -dominates all but $n/10k$ vertices in G , then $\gamma_k(G) \leq n/10k + 40k \leq n/6k$, so $\gamma_k(G)\gamma_k(\overline{G}) < n$, a contradiction. Therefore, at least $n/10k > 2^{40k}(k - 1)$ vertices have at most $k - 1$ neighbors in G in the set K , so there are $B \subseteq K$ and $A \subseteq V(G) - K$ with $|B| < k$ and $|A| = k$ such that $N_G(v) \cap K = B$ for all $v \in A$. Then A, B , and the v_i 's are a k -dominating set of \overline{G} with size at most $3k$, so we conclude $\gamma_k(\overline{G}) \leq 3k$.

If $k = 2$, first apply Lemma 3.3 to see $\gamma_2(\overline{G}) \leq 4$. If $\gamma_2(G)\gamma_2(\overline{G}) > 3(n - 1)$, then by Lemma 3.6 with $w = 2$ we know that G has at least $(n - 9)/12 \geq 8$ vertices of degree at most 1, so \mathcal{N}_G contains an $(r, 3)$ - Δ -system \mathcal{A} with r either 0 or 1. By Proposition 5.3, $\gamma_2(\overline{G}) \leq 3$. If $\gamma_2(G) \leq n - 1$ we are done. If $\gamma_2(G) = n$, then G has two isolated vertices or an isolated edge, so $\gamma_2(\overline{G}) = 2$. In either case, the bound holds.

Now suppose that $k \geq 3$. If Inequality 2 does not hold, then $\gamma_k(G) > 19n/36 \geq n/2 + 2(2k)^{4k-2}$. Then, by Lemma 3.6 with $w = 1$, G has at

least $2(2k)^{4k-2}$ vertices of degree less than $2k - 1$. By Proposition 5.2, the set system \mathcal{N}_G contains an $(r, k + 1)$ - Δ -system with $r < 2k - 1$; let \mathcal{A} be such a Δ -system with minimum r . Then $\gamma_k(\overline{G}) \leq r + k$ by Lemma 5.3. If $\gamma_k(G) > wn/(w+1) + n/10k$ with $w = \lfloor r/(r-k+1) \rfloor$, then by Lemma 3.6 G has at least $n/10k > 2(2k)^{4k-2}$ vertices of degree less than $r' = (w+1)k/w - 1$. From Lemma 4.1 we have $r' < r$, so \mathcal{N}_G contains a $(q, k + 1)$ - Δ -system with $q \leq r' < r$, a contradiction. Therefore,

$$\gamma_k(G)\gamma_k(\overline{G}) \leq \left(\frac{wn}{w+1} + \frac{n}{10k} \right) (k+r).$$

If $r \geq k$, we can apply Proposition 4.2 to see the right hand side is

$$\leq n(2k - 4/3) + 3kn/10k < (2k - 1)(n - k + 1).$$

We are left only with the case that $r < k$. Let K be the kernel of \mathcal{A} . Each vertex of K has degree at least $k + 1$ in G , so $V(G) - K$ is a k -dominating set of G , which tells us $\gamma_k(G) \leq n - |K|$. If $|K| = r$, Proposition 5.3 tells us $\gamma_k(\overline{G}) \leq |K| + k$, so $\gamma_k(G)\gamma_k(\overline{G}) \leq (k + |K|)(n - |K|) \leq (2k - 1)(n - k + 1)$. If $|K| < r$, then Proposition 5.3 tells us $\gamma_k(\overline{G}) \leq k + 1 + |K|$. Since $|K|$ must be at most $k - 2$, $\gamma_k(G)\gamma_k(\overline{G}) \leq (k + 1 + |K|)(n - |K|) \leq (2k - 1)(n - k + 2)$ and the desired bound holds. \square

Theorem 6.5 *Let $k \geq 2$ be an integer. Let G be a graph with order $n \geq 72(40k)^{80k}$. Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2).$$

Proof. If every $(2k - 1)$ -set $X \subseteq V(G)$ satisfies that $N_G^k(X)$ k -dominates all but at most $2k - 2$ vertices of $V(G) - N_G^k[X]$, then we are done by Lemma 6.3 (noting that n is at least $10^6 k^2$).

Otherwise, let X be an exceptional $(2k - 1)$ -set, and let Y be the vertices of $V(G) - N_G^k[X]$ not k -dominated in G by $N_G^k(X)$. Let W be the set of vertices of $V(G) - X$ not k -dominated by Y in \overline{G} . Then

$$(|W| - k + 1)|Y| \leq |Y||W| - |E_G(Y, W)| = |E_{\overline{G}}(Y, W)| \leq (k - 1)|W|,$$

so $|W| \leq |Y|(k - 1)/(|Y| - k + 1) \leq 2k - 2$. Since $W \cup X \cup Y$ is a k -dominating set in \overline{G} , we have $\gamma_k(\overline{G}) \leq |W| + |X| + |Y| \leq 6k - 5$. Now apply Lemma 6.4; this gives the Theorem. \square

7 Nearly Regular Graphs

It may seem that to maximize $\gamma_k(G)\gamma_k(\overline{G})$, we would want to make all the degrees of G small in order to make $\gamma_k(G)$ large. Perhaps surprisingly, this is not the case. Indeed, if $\gamma_k(G)\gamma_k(\overline{G})$ is to be large, one of the ratios $\Delta(G)/\delta(G)$ and $\Delta(\overline{G})/\delta(\overline{G})$ must be large as well; this is the content of Theorem 7.1.

Theorem 7.1 *Let $k \geq 2$ be an integer. Let G be a graph of order $n \geq 10^{20}k^{10}(\log k)^{10}$ such that $\rho = \max\{\Delta(G)/\delta(G), \Delta(\overline{G})/\delta(\overline{G})\} \leq n^{1/5}$. Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (k+1)n.$$

Furthermore, if G and \overline{G} have minimum degree at least k , then

$$\gamma_k(G)\gamma_k(\overline{G}) \leq kn.$$

Both bounds above are sharp. If G is a $(k-1)$ -regular K_k -free graph with at least k^3 vertices, then we have $\gamma_k(G) = n$ and $\gamma_k(\overline{G}) \geq k+1$ (if $\gamma_k(\overline{G}) = k$, the vertices of a minimum k -dominating set in \overline{G} would form a K_k in G), so G witnesses the sharpness of the first bound. If we insist G and \overline{G} have minimum degree at least k , we take G to be a disjoint union of (at least two) K_{k+1} 's. Then any minimum k -dominating set of G contains exactly k vertices from each component, so $\gamma_k(G) = kn(G)/(k+1)$. It is easy to see $\gamma_k(\overline{G}) = k+1$, so G demonstrates the sharpness of the second bound.

Proof. By Corollary 3.5, without loss of generality, $\gamma_k(\overline{G}) \leq (2k-1)\lceil \log n \rceil$.

Let M denote the set of vertices v with $d_G(v) \leq n^{1/5}$. Suppose $|M| \leq n^{4/5}$. Since $n^{1/5} \geq 7k \log k$, by Lemma 3.1,

$$\gamma_k(G) \leq \left(\frac{k \log n}{5n^{1/5}} + \frac{k(2e \log n)^{k-1}}{5^{k-1}n^{k/5}} \right) n + n^{4/5}.$$

Since n is large, we have $2e \log n \leq n^{1/5}$, so

$$\gamma_k(G) \leq n^{4/5}(k \log n/4 + k/4 + 1).$$

But then

$$\gamma_k(G)\gamma_k(\overline{G}) \leq n^{4/5}(k \log n)^2 \leq n.$$

We may therefore assume $|M| \geq n^{4/5}$. Since there are at most $\rho n^{2/5} \leq n^{3/5}$ vertices of M within distance 2 in G from any vertex of M , we can

find a set of $n^{1/5} \geq k + 1$ vertices pairwise at distance at least 3 in G . But $k + 1$ of these vertices are a k -dominating set in \overline{G} , so $\gamma_k(\overline{G}) \leq k + 1$. This immediately implies the first bound. If \overline{G} has minimum degree at least k , then $\gamma_k(\overline{G}) \leq kn/(k + 1)$ by Theorem 1.1, which gives us the second. \square

8 Game Domination

The game domination number is defined in terms of the following procedure. Two players, D (dominator) and A (avoider or adversary), alternately orient edges of a graph G , starting with D. The goal of D is to make the domination number of the resulting directed graph as small as possible, whereas A tries to make it large. We define $\gamma_g(G)$ to be the domination number of the resulting graph if both D and A play their optimal strategy.

Game domination was introduced in [1], where γ_g was computed for several classes of graphs and a number of extremal problems were solved. For example, the following Nordhaus-Gaddum bound was proven.

Theorem 8.1 ([1]) *For any graph G on n vertices, $\gamma_g(G) + \gamma_g(\overline{G}) \leq n + 2$.*

Theorem 8.1 is sharp if $G = K_n$. The authors of conjectured this could be improved.

Conjecture 1 ([1]) *If G and \overline{G} are connected graphs with order n , then*

$$\gamma_g(G) + \gamma_g(\overline{G}) \leq 2n/3 + 3.$$

In the remainder of this section, we prove Conjecture 1 if n is large.

We first recall some useful results.

Theorem 8.2 ([1]) *If G is a connected graph on $n \geq 2$ vertices, then $\gamma_g(G) \leq 2n/3$. Furthermore, if G has no isolated vertices, we also have $\gamma_g(G) \leq 2n/3$.*

If G is a graph with minimum degree at least 2 and order n , then $\gamma_g(G) \leq n/2$.

Suppose D and A are playing their domination game on a graph G , and let X be a 2-dominating set of G . As soon as A orients an edge from $v \in$

$V(G) - X$ into X , D , on his next turn, can orient an edge from X into v . In this way, X dominates the oriented digraph D and A create. Therefore, as noted in [1], $\gamma_g(G) \leq \gamma_2(G)$ for all graphs G .

We can sometimes do better than this, though. Define a *nearly-2-dominating set* of G to be a set $X \subset V(G)$ such that X dominates G and 2-dominates all but at most one vertex of G . Let $\gamma_2^*(G)$ be the size of the smallest nearly-2-dominating set of G . Then, if X is a nearly-2-dominating set in G and v is the vertex not 2-dominated by X (if it exists), then D can orient an edge from X to v on his first turn, and then orient edges from X to the other vertices of $V(G) - X$ in response to A . This observation gives our next Proposition.

Proposition 8.3 *For all graphs G , $\gamma_g(G) \leq \gamma_2^*(G) \leq \gamma_2(G)$.*

This Proposition is the bridge from our earlier results to game domination.

Theorem 8.4 *For any n -vertex graph G with $n \geq 4,000,000$,*

$$\gamma_g(G)\gamma_g(\overline{G}) \leq 2n + 12.$$

Proof. By Theorem 6.3 and Proposition 8.3, we are done unless (without loss of generality) there is some 3-set X such that $N_G^2(X)$ does not 2-dominate two vertices v_1 and v_2 . Let y_i be the neighbor in G of v_i in the set $N_G^2(X)$ for $i = 1, 2$, if it exists. Then $X \cup \{v_1, v_2, y_1, y_2\}$ is a 2-dominating set of \overline{G} , so $\gamma_2(\overline{G}) \leq 7$.

By Lemma 3.3 and Proposition 8.3, we may assume that $\gamma_2(\overline{G}) \leq 4$.

Suppose G has three vertices v_1, v_2 , and v_3 of degree at most 1, and let x_i be the neighbor of v_i (if it exists) for $i = 1, 2, 3$. Then $\{v_1, v_2, v_3\}$ is a 2-dominating set of \overline{G} unless, without loss of generality, $y_1 = y_2$. In that case, $\{v_1, v_2, y_1\}$ is a 2-dominating set of \overline{G} . In either case, $\gamma_g(\overline{G}) \leq 3$. By Theorem 8.2, if G has no isolated vertex, $\gamma_g(G) \leq 2n/3$, implying $\gamma_g(G)\gamma_g(\overline{G}) \leq 2n$. Suppose, then, that v_1 is isolated. Then $\{v_1, v_2\}$ is a nearly-2-dominating set of \overline{G} , so $\gamma_g(G)\gamma_g(\overline{G}) \leq 2\gamma_g(G) \leq 2n$.

Therefore we may assume that G , and by symmetry \overline{G} , has at most two vertices of degree at most 1. It follows from Theorem 8.2 that $\gamma_g(G)$ and $\gamma_g(\overline{G})$ are each at most $n/2 + 3$ (add at most 3 edges to G which are incident to at most 4 vertices to increase the minimum degree to 2, then consider deleting those vertices). From the second paragraph, one of these should also be at most 4. Therefore $\gamma_g(G)\gamma_g(\overline{G}) \leq 2n + 12$, so we are done. \square

Theorem 8.5 *Let G be a graph with $n \geq 4,000,000$ vertices such that G and \overline{G} are connected. Then $\gamma_g(G) + \gamma_g(\overline{G}) \leq 2n/3 + 3$. That is, Conjecture 1 holds.*

Proof. Maximize $x + y$ ($x = \gamma_g(G)$, $y = \gamma_g(\overline{G})$) subject to $x, y \leq 2n/3$ (by Theorem 8.2) and $xy \leq 2n + 12$ (by Theorem 8.4). \square

Notice Theorem 8.4 can also be used to prove Theorem 8.1.

9 Concluding Remarks and Open Problems

The Nordhaus-Gaddum bounds given in this paper are all sharp in some sense. The bound given in Theorem 6.5, namely that $\gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 2)$ was shown to be sharp if k is odd (and greater than 1). It seems that when k is even the stronger bound $\gamma_k(G)\gamma_k(\overline{G}) \leq (2k - 1)(n - k + 1)$ should hold, so that $K_{k-1, n-k+1}$ is extremal.

It would of course be interesting to have a better bound on $n_0(k)$ in Theorem 6.5. The bound is probably polynomial in k , possibly even degree 3. To do this would require a new method involving more structural considerations than Δ -systems.

One interesting direction could be to give Nordhaus-Gaddum bounds in terms of other parameters than just the order. This leads to the following question: what is the maximum value, in terms of $n(G)$ and d , of $\gamma_k(G)\gamma_k(\overline{G})$ if $\delta(G), \delta(\overline{G}) \geq d$? The special case $d = k$ would be the most interesting (this was partially answered in Theorem 7.1).

Finally, it would be most interesting to find extremal bounds on $\gamma_k(G)$, for example, in terms of $\delta(G)$. Good bounds were by Caro and Roditty (see Lemma 3.6) in the case when $\delta(G) \leq 2k - 1$. Above that, though, not much is known.

References

- [1] N. Alon, J. Balogh, B. Bollobás, and T. Szabó, *The Game Domination Number*, Discrete Math. 256 (2002) 23-33.
- [2] E.J. Cockayne, B. Gamble, and B. Shepherd, *An Upper Bound for the k -Domination Number of a Graph*, J. Graph Theory 9 (1985) 533-534.

- [3] E.W. Chambers, W. Kinnersley, N. Prince, and D.B. West, *Extremal Problems for Roman Domination*, submitted.
- [4] E.W. Chambers, W. Kinnersley, and N. Prince, *Mobile Eternal Security in Graphs*, submitted.
- [5] Y. Caro and Y. Roditty, *A Note on the k -Domination Number of a Graph*, Internat. J. Math. Math. Sci. 13 (1990) 1, 205-206.
- [6] P. Erdős, R. Faudree, A. Gyárfás, and R.H. Schelp, *Domination in Colored Complete Graphs*, J. Graph Theory 13 (1989) 6, 713-718.
- [7] P. Erdős and R. Rado, *Intersection Theorems for Systems of Sets*, J. London Math. Soc. 35 (1960) 85-90.
- [8] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Monographs and Textbooks in Pure and Applied Mathematics 208, Marcel Dekker, Inc., New York, 1998.
- [9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs: Advanced Topics*, Monographs and Textbooks in Pure and Applied Mathematics 209, Marcel Dekker, Inc., New York, 1998.
- [10] F. Jaeger and C. Payan, *Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple*, C. R. Acad. Sci. Paris Sér. A 274 (1972) 728-730.
- [11] A.V. Kostochka, *Extremal Problems on Δ -Systems*, Numbers, Information, and Complexity pp. 143-150, Kluwer Acad. Publ., Boston, 2000.
- [12] D.B. West. *Introduction to Graph Theory, 2nd ed.* Prentice Hall, Upper Saddle River, NJ, 2001.