Mobile Eternal Security in Graphs

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September 14, 2006

Abstract

We consider the problem of securing a graph against a sequence of vertex attacks by placing guards on its vertices. The guards are "mobile" in that, after any attack, each guard can move to any neighbor of its current location, but one guard must move to the attacked vertex.

We determine sharp upper bounds, in terms of the order of the graph, on the minimum number of guards necessary for connected graphs and graphs with minimum degree at least 2. We also derive sharp Nordhaus-Gaddum type bounds.

There are many different notions for the "security" of a structure, which depend heavily on how the structure is attacked and how it can be protected. When the structure in question is a graph, several attack/protection schemes have already been studied [2, 3, 4, 5, 6].

In this paper, we study a newer protection scheme introduced in [7]. In this model, mobile guards are placed on the vertices of a graph. The guards can move to any neighbor of their current location in any time step.

[*]Research partially supported by an NSF Graduate Research Fellowship and by NSF grant DMS-0528086.

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The vertices of the graph are attacked one at a time. After each attack, the guards must reconfigure themselves so that some guard moves to the attacked vertex. If the guards can always shift in such a way as to be able to respond to any (possibly infinite) sequence of attacks, we say they "protect" the graph G. The *mobile eternal security number* of G, denoted $\sigma_m(G)$, is the least number of guards needed to protect G (this parameter is also called the *eternal m-security number* in [7]).

More formally, we say a function $f : [k] \to V(G)$ is $(k, 0)$-secure if the set $\{f(i) \,|\, i \in [k]\}$ dominates G. If $n \geq 1$, we say f is (k, n)-secure if, for any $v \in V(G)$, there is a $(k, n-1)$-secure function f' such that $f'(i) \in N_G[f(i)]$ for all $i \in [k]$ and $v = f'(j)$ for some j. Then $\sigma_m(G)$ is the minimum k for which G has a function f which is (k, n)-secure for all $n \in \mathbb{N}$. In this paper, we shall argue in the less formal language of guards protecting G rather than (k, n)-secure functions.

Note that a (k, n)-secure function need not be injective. This means we are allowed to station more than one guard at a vertex at one time. The arguments for Theorems 1, 3, and 4 can be modified slightly where needed to work even without placing multiple guards at a vertex, although this is not the case for the proof of Theorem 2. We have no examples, however, which suggest Theorem 2 becomes untrue if we insist only one guard be placed at a vertex at any given time.

For trivial bounds on $\sigma_m(G)$, note that at any time, the set of locations of guards must be a dominating set, so $\sigma_m(G) \geq \gamma(G)$ for all graphs G. In fact, if some vertex $v \in V(G)$ is in no minimum dominating set, then $\sigma_m(G) \geq \gamma(G) + 1$ (consider the guard locations after v is attacked). This small improvement will be helpful when looking at certain small graphs later.

For upper bounds, we note that $\sigma_m(G) \leq \Theta(G)$ where Θ is the clique cover number, since one guard can protect an entire clique. In terms of domination parameters, consider $\gamma_c(G)$, the minimum size of a connected dominating set. It is shown in [7] that $\sigma_m(G) \leq \gamma_c(G) + 1$. If G is connected, then $\gamma_c(G) \leq 3\gamma(G) - 2$, so $\sigma_m(G) \leq 3\gamma(G) - 1$. This upper bound can be improved to $2\gamma(G)$ by noting that $\sigma_m(G)$ is at most $\gamma_R(G)$ [7], the Roman domination number of G, and that $\gamma_R(G) \leq 2\gamma(G)$ for all G.

In this paper, we bound $\sigma_m(G)$ in terms of the order of G. In Section 1, we derive an upper bound of $\lceil n/2 \rceil$ on the mobile eternal security number of a connected n-vertex graph. In Section 2, we prove several lemmas needed in Section 3, where we show that if G has minimum degree at least 2, then $\sigma_m(G) \leq \lceil n(G)/2 \rceil - 1$. Finally, in Section 4, we find Nordhaus-Gaddum type bounds on the quantities $\sigma_m(G) + \sigma_m(\overline{G})$ and $\sigma_m(G)\sigma_m(\overline{G})$, where $\overline{G}$ is the complement of G. All these bounds are sharp.

Before we begin, let us fix notation. We write $n(G)$ for the number of vertices of a graph G and $\delta(G)$ for its minimum degree. The (open) neighborhood of v in the graph G is written $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. If $A \subseteq V(G)$, then the subgraph of G induced by A is $G[A]$. If G and H are graphs, then $G + H$ denotes their disjoint union. C_n and P_n are the cycle and path on n vertices, respectively. For further background and notation, see [8].

1 Connected Graphs

The goal of this section is to bound the mobile eternal security number of a connected graph in terms of its order. In [7] it is shown that $\sigma_m(P_n) = \lceil n/2 \rceil$, since at time i we can attack the vertex at distance $2i-2$ from one endpoint of the path; each attack requires a different guard. This is, in fact, the largest σ_m can be on connected order n graphs.

Theorem 1 *Let G be a connected graph of order n. Then $\sigma_m(G) \leq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Since adding edges cannot increase σ_m, we may assume G is a tree. We proceed by induction on n; the base cases $n = 1, 2$ are trivial.

Consider $n > 2$. Let u be an endpoint of a longest path of G, and let v be the neighbor of u.

If $d_G(v) = 2$, let $G' = G - u - v$. By the induction hypothesis, G' can be protected with at most $\lceil n/2 \rceil - 1$ guards. Then we can place one additional guard on v to protect u and v by alternating between them as necessary, so in this case we are finished.

Now suppose $d_G(v) \geq 3$. Let S be the set of leaves adjacent to v. By the induction hypothesis, $G' = G - S$ can be protected with at most $\lceil (n - |S|)/2 \rceil < \lceil n/2 \rceil$ guards. Now add a guard on v to protect S as follows: whenever a vertex in S, say s, comes under attack, move the guard from v to s and move another guard from a vertex of G' to v (this is possible by considering this as an attack on v in G'). At the first time step when s is not attacked, move the guard from it to v. If some other vertex $t \in S$ was attacked in this step, move a guard from v to t (we now consider the guard at t to be the one protecting S and the one at v to be protecting $V(G')$). In this way, we always keep one guard protecting S and $\sigma_m(G')$ guards protecting

$V(G')$, so the induction goes through. □

Applying Theorem 1 to the components of an arbitrary graph shows $\sigma_m(G) \le (n(G) + c_{\text{odd}}(G))/2$, where c_{odd} counts the odd-order components of G. Then, if G has no isolated vertices, we see $\sigma_m(G) \le 2n/3$.

2 Some Lemmas

In this section, we derive several lemmas we will need in the next section. Two of these determine the mobile eternal security numbers of some special graphs which arise in the proof of Theorem 2.

Note that for a cycle C_n, the mobile eternal security number equals the domination number, $\lceil n/3 \rceil$, because we may place guards on a dominating set and rotate them to respond to attacks. This protection strategy generalizes well.

A *walk* in a graph G is a sequence of vertices $w_0, w_1, \ldots, w_k$ such that $w_{i-1}w_i \in E(G)$ for all $1 \le i \le k$. The length of the walk is k. A walk in a graph G is *closed* if its endpoints coincide and *Hamiltonian* if it includes every vertex of G.

Lemma 1 *Suppose G has a closed Hamiltonian walk with length k. Then $\sigma_m(G) \le \lceil k/3 \rceil$.*

Proof. Let $w_0, w_1, \ldots, w_k$ be a closed Hamiltonian walk of G. Let $f(i)$ be the vertex of G for which $f(i) = w_i$. Let $x_1, x_2, \ldots, x_k$ be the vertices of C_k in cyclic order. Note that, for all $i \in [k]$, $f(i)f(i+1)$ is an edge of G (addition done modulo k).

We define a protection strategy for G as follows. View an attack on vertex $v \in V(G)$ as an attack on x_j in C_k, where $j = f^{-1}(v)$. Whenever an optimal protection strategy for C_k, responding to this attack, moves a guard from x_p to x_q, move a guard from $f(p)$ to $f(q)$. This strategy protects G with $\sigma_m(C_k) = \lceil k/3 \rceil$ guards, so we are done. □

We would like to exploit the ease with which cycles are protected for graphs which are cycle-like. The next lemma describes a scenario where we can.

Proposition 1 *Let G be a graph obtained by intersecting a graph H with C_{3k+1} in a single vertex v. Then $\sigma_m(G) \le k + \sigma_m(H)$.*

Proof. Let the vertices of the cycle be $v, v_1, \ldots, v_{3k}$ in cyclic order. By the i-vertices we mean the set $\{v_{3j+i} \mid 0 \leq j \leq k-1\}$.

Initially place a guard on each 2-vertex and on an initial placement for H. Refer to the guards on $V(H)$ as the H-guards and the rest as "cycle guards". At each time step, we will ensure that the guards protect G and that the cycle guards are on the i-vertices for some $i \in \{0, 1, 2\}$.

If a vertex of H is attacked, move an H-guard to it and move the cycle guards to the 2-vertices. If a vertex not in H is attacked, say an i-vertex, rotate the guards to cover all the i-vertices and move an H-guard to v (by viewing this as an attack in H on v). This always protects G unless it causes some cycle guard g to move from v_j to v. In this case, notice the cycle guards were not on the 2-vertices, so the previous attack was not on a vertex of H, and hence there is a guard at v; move this guard to v_{3k+1-j} and switch the roles of g and the H-guard on v. In any case, G is protected and the cycle guards cover the i-vertices for some i, so we may continue inductively to protect G. □

Usually we will apply Proposition 1 with H a clique, most often taking $H = K_2$.

Finally, we must determine σ_m of a very specific graph.

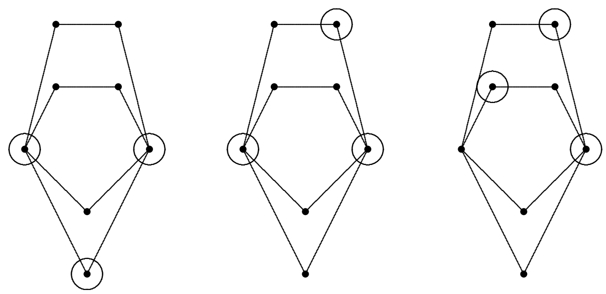
Proposition 2 *Let G be the graph obtained by intersecting two C_5's in a pair of nonadjacent vertices. Then $\sigma_m(G) = 3$.*

Proof. Note $\sigma_m(G) \geq 3$ because any dominating set containing a degree 2 vertex must have size at least 3.

For the upper bound, consider the three (isomorphism classes of) guard locations pictured in Figure 1. Initially place the guards in State 1. Let the vertices of degree 4 be u and v, with u on the left. We consider three cases as to the position of the guards at the time of an attack.

Consider being attacked in State 1. If u, v, or one of their common neighbors is attacked, shift the guards to protect it in such a way as to end in State 1. Otherwise, shift the guards to arrive in State 2.

Now consider being attacked in State 2. If the unique vertex distinct from u and protected only by u is attacked, shift to State 3. If a common neighbor of u and v is attacked, shift to State 1. Otherwise, we can shift to remain in State 2.



State 1 State 2 State 3

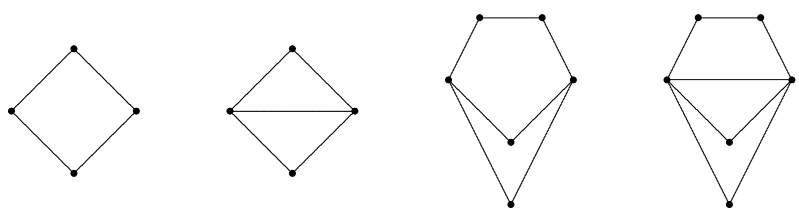
Figure 1

Finally, consider being attacked in State 3. If u, v, or one of their common neighbors is attacked, shift to State 1. If some other unguarded vertex is attacked, shift to State 2.

This strategy protects G with 3 guards, so the Proposition is proved. □

3 Minimum Degree 2

Now we are ready to bound σ_m for connected graphs with minimum degree at least 2. Considering two disjoint triangles with a path between them shows σ_m can be as large as $\lceil n(G)/2 \rceil - 1$ on these graphs. With the exception of a small family $\mathcal{C}$ (Figure 2) of counterexamples, this is the extremal value.



C_4 C_4' E_6 E_6'

Figure 2: The family $\mathcal{C}$

Note that $\sigma_m(C_4') \geq \sigma_m(C_4) \geq \gamma(C_4) = 2$ and in E_6 and E_6', a common neighbor of the vertices of maximum degree is in no minimum dominating set, so $\sigma_m(E_6') \geq \sigma_m(E_6) \geq \gamma(E_6) + 1 = 3$.

We will need the following proposition about the members of $\mathcal{C}$.

Proposition 3 *Let G be obtained by adding a pendant edge to an element of $\mathcal{C}$. Then $\sigma_m(G) \leq (n(G) - 1)/2$.*

Proof. We can find a closed Hamiltonian walk of G with length $\frac{3}{2}(n(G) - 1)$, so we are done by Proposition 1. □

We are now ready for the Theorem.

Theorem 2 *Let G be a graph of order n with $\delta(G) \geq 2$ not in the family $\mathcal{C}$. Then $\sigma_m(G) \leq \lceil \frac{n}{2} \rceil - 1$.*

The strategy of the proof will be to apply four simple reductions to G until the cycles of G have a special structure which facilitates induction. The proof would be considerably shorter were it not for the graphs in $\mathcal{C}$, since we must separately consider when each reduction or inductive step encounters one of them.

Proof. Suppose the Theorem is false. Let G be a counterexample with the fewest vertices, and subject to that, with the fewest edges. In particular, G is edge-minimal with respect to the conditions that it is connected and has minimum degree at least 2. Notice $n(G) \geq 4$. Then we claim that
(1) every cycle of G is chordless,
(2) no two vertices of degree at least 3 in G appear consecutively along a cycle,
(3) no two vertices of degree 2 are adjacent,
(4) no vertex of degree at least 4 in G lies on a cycle.

Statements (1) and (2) follow immediately from edge-minimality.

For (3), suppose G has two adjacent vertices u and v of degree 2. Suppose u and v have a common neighbor w. If $d_G(w) \geq 4$, place a guard on $\{u, v\}$, and protect $G - u - v$ inductively, unless $G - u - v \in \mathcal{C}$. In this case, if $n(G - u - v) = 4$, we apply Proposition 1 to see $\sigma_m(G) = 2$, and if $n(G - u - v) = 6$, by inspection, $\sigma_m(G) = 3$. Therefore assume $d_G(w) = 3$.

Let x be the vertex of degree at least 3 in G with minimum distance to w. Let G' be obtained from G by deleting $\{u, v, w\}$ and all internal vertices along the shortest w, x-path (call these vertices P). If $G' \notin \mathcal{C}$, then $\sigma_m(G') < n(G')/2$ by induction. Since $\sigma_m(G[P]) \leq (|P|+1)/2$ and $\sigma_m(G[u, v, w]) = 1$, we are done. If $G' \in \mathcal{C}$ and $P \neq \emptyset$, then let y be the neighbor in P of x. Then $\sigma_m(G' \cup \{y\}) < (n(G') + 1)/2$ by Proposition 3, and, as above, $\sigma_m(G[(P - z) \cup \{u, v, w\}]) \leq (|P| + 2)/2$, so we are done. Therefore, we assume P is empty. But then G contains as a spanning subgraph the disjoint union of an edge and an element of $\mathcal{C}$ with a pendant edge added, and we see that $\sigma_m(G) < n(G)/2$. Thus we may assume u and v have no common neighbor.

Now consider when the neighbors in G of u and v, say t and w, respectively, are distinct. If t and w are nonadjacent, then let G' be $G-u-v$ with edge tw added. G' is connected with minimum degree at least 2, so if it is not in $\mathcal{C}$, we can protect it with fewer than $n(G')/2$ guards, and we can place a single guard on $\{u, v\}$. For a protection strategy of G, if (without loss of generality) u is attacked, move the guard on $\{u, v\}$ to u and a guard from G' to w. Respond to an attack on $V(G')$ by playing an optimal strategy in G', and if a G'-guard is then stationed at t, move the $\{u, v\}$-guard to v. In this way, if, responding to a subsequent attack on w, an optimal strategy for G' moves the guard from t to w, we can simulate this by moving the guard on t to u and the one on v to w. This protects G with fewer than $n(G)/2$ guards. We may therefore assume $G' \in \mathcal{C}$. In this case, G is one of the graphs in Figure 3, with the possible addition of an edge. Now C_6 needs only 2 guards, the middle three graphs need only 3 guards by Proposition 1 (they contain C_7 with a pendant edge), and the rightmost graph needs only 3 guards by Proposition 2, so all these graphs satisfy the conclusion of the Theorem.

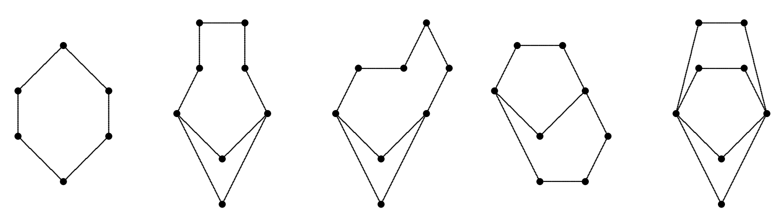


Figure 3

We may assume, then, that t and w are adjacent. If both have degree 2, G is C_4, contradicting our hypothesis. Consider when each has degree at least 3. Then $G - u - v$ is connected and has minimum degree at least 2, so we can proceed by induction unless $G' \in \mathcal{C}$. In this case, G contains as a spanning subgraph one of the graphs pictured in Figure 3 or 4. We have seen how to protect the those in Figure 3, and each graph in Figure 4 has a closed Hamiltonian walk with length 8 or 9, so we are done by Lemma 1.

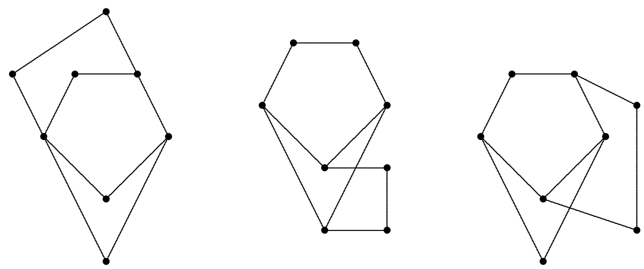


Figure 4

Therefore, we may assume, without loss of generality, that $d_G(t) = 2$, $d_G(w) \geq 3$, and $tw \in E(G)$. Then $S = \{t, u, v, w\}$ induces C_4. Let x be the vertex of degree at least 3 closest to w, and let P be the set of internal vertices of the shortest w, x-path. Let $G' = G - P - S$.

If G' is not in $\mathcal{C}$, then $\sigma_m(G') < n(G')/2$. If P has even size, we can cover it with $|P|/2$ guards and cover S with 2 guards. If $|P|$ is odd, and $x \in P$ is the vertex adjacent to w, then we can protect $S \cup \{x\}$ with 2 guards and the remaining vertices of P with $(|P| - 1)/2$ guards. In either case, $\sigma_m(G[P \cup S]) \leq (|P| \cup |S|)/2$, so we are done.

Assume, then, that $G' \in \mathcal{C}$. If $|P| \geq 2$, we can break G into three vertex-disjoint subgraphs, two of which induce an element of $\mathcal{C}$ with a pendant edge and the other induces a path. If $|P| = 1$, then we break G into two vertex-disjoint subgraphs, one of which is C_4 with a pendant edge and the other of which is connected. In either case, we are done by Proposition 3 and Theorem 1. If P is empty, we apply Propositions 3 and 1 to see $\sigma_m(G) < n(G)/2$. This establishes (3).

Finally, for (4), let v have degree at least 4 in G and suppose v lies on a cycle C. Let u and w be the neighbors of v in C. By (2), u and w have degree 2. If t and x are the neighbors in C of u and w (possibly $t = x$), respectively, then (3) tells us $d_G(t)$ and $d_G(x)$ are at least 3. Suppose some vertex y in $N_G(v) - \{u, w\}$ has degree at least 3. Then, by edge-minimality of G, vy is a cut-edge. Since $G - vy$ has minimum degree at least 2, every component

C satisfies $\sigma_m(C) \le n(C)/2$, so if either component of $G - vy$ is not in $\mathcal{C}$, we are done. If both components are in $\mathcal{C}$, we are done by inspection.

We may therefore assume every vertex of $N_G(v) - \{u, w\}$ has degree 2. Therefore, by (3), we may assume that each neighbor of a vertex in $N_G(v)$ has degree at least 3 in G. Let $G' = G - N_G[v]$. Each component of G' has minimum degree at least 2, so $\sigma_m(G') \le n(G')/2$. Since we only need 2 guards to protect the ≥ 5 vertices of $N_G[v]$, we see that $\sigma_m(G) < n(G)/2$, so we are done. This establishes the claim.

If G contains an odd cycle, (2) or (3) will be contradicted, so we may assume G is bipartite. Suppose C is a cycle in G of length $2k$ with vertices $v_1, v_2, \ldots, v_{2k}$ in order. By (4), every vertex of C has degree at most 3, so by (2) and (3), we may assume $d_G(v_i)$ is 3 if i is even and 2 if i is odd. Let y_i be the neighbor of v_{2i} in $G - V(C)$ and C_i be the component of $G - C$ containing y_i. If $d_G(y_i) \ge 3$, by edge-minimality of G, $C_i \in \mathcal{C}$. Then

$$\sigma_m(G) \le \sigma_m(G - C_i) + \sigma_m(C_i) \le \left\lceil \frac{n(G - C_i)}{2} \right\rceil - 1 + \frac{n(C_i)}{2} \le \left\lceil \frac{n(G)}{2} \right\rceil - 1$$

by induction. Hence we may assume all y_i's have degree 2.

Let $B \subset \{\, y_i \mid 1 \le i \le k \}$ contain exactly one vertex of each odd component of $G - V(C)$ and let $|B| = b$. Let $G_1 = G[C \cup B]$. Since every component of $G_2 = G - V(G_1)$ has even order, by Theorem 1, $\sigma_m(G_2) \le n(G_2)/2$. Now, G_1 has a closed Hamiltonian walk of length $2k + 2b$, so $\sigma_m(G_1)$ is at most $\lceil (2k + 2b)/3 \rceil$ by Lemma 1, which in turn is less than $n(G_1)/2 = (2k + b)/2$, implying $\sigma_m(G) < n(G)/2$, unless (k, b) is $(8, 4)$ or $(4, 2)$. In these cases, $\sigma_m(G_1) = n(G_1)/2$. Note that every C_i has minimum degree at least 2, otherwise (3) would be contradicted. If any C_i is not in $\mathcal{C}$, then $\sigma_m(G) < n(G)/2$. Therefore assume $C_i \in \mathcal{C}$. Let $H_i = G[C_i \cup \{y_i\}]$. Proposition 3 tells us $\sigma_m(H_i) < n(H_i)/2$. Since $\delta(G - H_i) \ge 2$, $\sigma_m(G - H_i) \le n(G - H_i)/2$, so we are done. □

4 Nordhaus-Gaddum Bounds

This section is devoted to deriving Nordhaus-Gaddum type bounds on σ_m for sums and products. We start with three quick lemmas.

Lemma 2 *If G has diameter 2, then $\sigma_m(G) \le \delta(G) + 2$.*

Proof. For any $v \in V(G)$, $N_G[v]$ is a connected dominating set of G. $\square$

Lemma 3 *If G is a graph with diameter at least 3, then $\sigma_m(\overline{G}) \leq 3$.*

Proof. Two vertices at distance at least 3 in G are a connected dominating set in $\overline{G}$. $\square$

Lemma 4 *If G has order n and at least 3 edges, $\sigma_m(G) \leq n - 2$.*

Proof. A graph with 3 edges has either two independent edges, a triangle, or a subgraph isomorphic to $K_{1,3}$. In the first two cases $\Theta(G) \leq n - 2$. In the third case we can put two guards on a copy of $K_{1,3}$ and one guard on every other vertex. $\square$

Before we derive the sum bound, we define some graphs. Let $\mathcal{F}_n = \{\overline{K_n}, K_2 + \overline{K_{n-2}}, K_{1,2} + \overline{K_{n-3}}\}$. Let the *bull* be the unique graph, up to isomorphism, with degree sequence 1,1,2,3,3. We claim the bull has mobile eternal security number 3. For the upper bound, its vertices can be covered with 3 cliques. For the lower bound, consecutive attacks on a vertex of degree 1, the vertex of degree 2, and the other vertex of degree 1 each require a separate guard.

Theorem 3 *Let G be a graph of order n. Then $\sigma_m(G) + \sigma_m(\overline{G}) \leq n + 1$, with equality iff G or $\overline{G}$ is a bull or in $\mathcal{F}_n$.*

Proof. It is easy to see that if G or $\overline{G}$ is in $\mathcal{F}_n$, then equality holds, so suppose $G, \overline{G} \notin \mathcal{F}_n$.

Suppose G is disconnected. Then $\sigma_m(\overline{G}) = 2$. By Lemma 4, if G has at least 3 edges, $\sigma_m(G) + \sigma_m(\overline{G}) \leq n$. Otherwise, since $G \notin \mathcal{F}_\backslash$, $G = 2K_2 \cup (n-4)K_1$, and the conclusion of the Theorem holds.

We may therefore assume G and, by symmetry, $\overline{G}$ are connected (so $n \geq 4$). If G and $\overline{G}$ have minimum degree at least 2, then by Theorem 2, $\sigma_m(G) + \sigma_m(\overline{G}) < n$, unless $G \in \{E_6, E_6'\}$. In this case, $\overline{G}$ contains the disjoint union of K_1 and C_4 with a pendant edge attached, so $\sigma_m(\overline{G}) \leq 3$, and so $\sigma_m(G) + \sigma_m(\overline{G}) \leq n(G)$ holds.

Assume, then, without loss of generality, that G has minimum degree equal to 1. Then $\sigma_m(\overline{G}) \leq 3$, so by Theorem 1,

$$\sigma_m(G) + \sigma_m(\overline{G}) \leq \frac{n+1}{2} + 3 < n + 1$$

if $n \geq 6$.

If $n = 4$, then $G = P_4$, so the conclusion holds, so suppose $n = 5$. If G is bipartite, $\sigma_m(\overline{G}) \leq \Theta(\overline{G}) = 2$, so we are done by Theorem 1 unless G and (by symmetry) $\overline{G}$ have odd cycles. If $G = C_5$ the conclusion holds, so these odd cycles must be triangles. We conclude that G is a bull, and we are done. □

For the product bound, we employ the method of [6].

Theorem 4 *Let G be a graph of order $n \geq 58$. Then $\sigma_m(G)\sigma_m(\overline{G}) \leq 2n-2$, with equality iff G or $\overline{G}$ is in $\mathcal{F}_n$.*

We note the bound on n above can probably be improved, but it cannot be entirely omitted; when G is the bull, $\sigma_m(G)\sigma_m(\overline{G}) = 9 > 2n(G) - 2$.

Proof. Suppose that G is disconnected. Then $\sigma_m(\overline{G}) \leq 2$, and since $\sigma_m(G) < n$ unless $G = \overline{K_n}$, the conclusion holds for G. Furthermore, if $\sigma_m(G)\sigma_m(\overline{G}) = 2n-2$, then $\sigma_m(G) = n-1$, so by Lemma 4, G has at most 2 edges. If G has two independent edges, $\sigma_m(G) < n-1$, so we conclude $G \in \mathcal{F}_n$.

We may assume, therefore, that G and, by symmetry, $\overline{G}$ are connected. Suppose G has diameter at least 3. Then $\sigma_m(\overline{G}) \leq 3$ by Corollary 3, and by Theorem 1, $\sigma_m(G) \leq (n+1)/2$, so $\sigma_m(G)\sigma_m(\overline{G}) < 2n-2$.

Therefore, we need only consider when G and, by symmetry, $\overline{G}$ have diameter 2. Assume, without loss of generality, that $\delta(G) \leq (n-1)/2$. Let v be a vertex of minimum degree in G.

Since G has diameter 2, $N_G(v)$ dominates $V(G) - N_G[v]$. Define sets $A_1, \ldots, A_q$ and A^* as follows: let A_i be a minimal dominating subset of $N_G(v) - \bigcup_{j=0}^{i-1} A_j$ if one exists, otherwise set $q = i-1$ and let $A^* = N_G(v) - \bigcup_{j=0}^{q} A_j$. Among all such partitions of $N(v)$, choose the one that maximizes q.

Now, v along with the smallest A_i is a connected dominating set of G, so $\sigma_m(G) \leq \delta(G)/q + 2$. Since each A_i was a minimal dominating set in G, there must be a vertex $x_i \in V(G) - N_G[v]$ which dominates all but one vertex, say a_i, of A_i. Also, since A^* did not dominate $V(G) - N_G[v]$ in G, there is a vertex x^* which dominates A^* in $\overline{G}$. Now, v, x^*, and the x_i's are a connected dominating set of $V(G) - \{a_1, \ldots, a_q\}$ in $\overline{G}$ of size $q+2$, so we can protect $V(G) - \{a_1, \ldots, a_q\}$ with $q+3$ guards, and putting an additional guard on each a_i shows $\sigma_m(\overline{G}) \leq 2q+3$.

Therefore

$$\sigma_m(G)\sigma_m(\overline{G}) \leq \left(\frac{\delta(G)}{q} + 2\right)(2q+3) \leq 5\delta(G) + 10, \tag{1}$$

achieved when $q = 1$. Assume that $q = 1$. Then, by the Equation 1, $\sigma_m(G)\sigma_m(\overline{G})$ is less than $2n-2$ when $\delta(G) < (2n-12)/5$, so we may assume that G has minimum degree at least $(2n-12)/5 \geq 9n/25 - 1$. Since $q = 1$, we may assume $\sigma_m(\overline{G}) = 2q+3 = 5$. We will be done if we can show in fact that $\sigma_m(G) < (2n-2)/5$. We use the following Lemma from [1] (page 4, Theorem 1.2.2):

Lemma 5 *If G has minimum degree at least k, then $\gamma(G) \leq \frac{1+\ln(k+1)}{k+1} n(G)$.*

Therefore, letting $k = 9n/25 - 1$ and using the fact that $\sigma_m(G) \leq 2\gamma(G)$, we have

$$\sigma_m(G) \leq \frac{50}{9}(1 + \ln(9n/25)) < \frac{2n-3}{5}$$

when $n > 56$. We may therefore assume $q > 1$.

Suppose now that $q > \delta(G)/3$, so that some A_i has size at most 2. Then $\gamma_c(G) \leq 3$, so $\sigma_m(G) \leq 4$, and we will be done if $\sigma_m(\overline{G}) \leq (2n-3)/4$. By Lemma 2, this happens when $\delta(\overline{G}) \leq (2n-11)/4$, so we may assume $\delta(\overline{G}) \geq (n-5)/2$. Suppose some vertex x is adjacent to at least 7 vertices of $V(G) - N_{\overline{G}}[v]$. Then putting two guards each on $N_{\overline{G}}[v]$ and $N_{\overline{G}}[x]$ and one guard on every other vertex shows

$$\sigma_m(\overline{G}) \leq 4 + (n - (\delta(\overline{G}) + 1) - 7) \leq \frac{n-7}{2},$$

so we are done. If no vertex in $N_{\overline{G}}(v)$ has seven neighbors in $V(G) - N_{\overline{G}}[v]$, then there are at most $6|N_{\overline{G}}(v)|$ edges between $N_{\overline{G}}(v)$ and $V(G) - N_{\overline{G}}[v]$. Since $|V(G) - N_{\overline{G}}[v]| \leq |N_{\overline{G}}(v)|$, some vertex $x \in N_{\overline{G}}[v]$ has at most 6 neighbors in $N_{\overline{G}}[v]$, so has at least $\delta(G) - 6 \geq 7$ neighbors in $V(G) - N_{\overline{G}}[v]$, so we are done. Therefore we may assume that $q \leq \delta(G)/3$.

We have then that $2 \leq q \leq \delta(G)/3$. In this range, $(\delta(G)/q + 2)(2q+3) \leq 7\delta(G)/2 + 14$, which is less than $2n-2$ if $n > 57$. This completes the proof. □

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