All problems must have non-calculator work for full credit.

1) (10 points) Find the $PA = LU$ factorization for the matrix $A$ below. Then find bases for the row space, column space, and nullspace of $A$ (don’t bother with the left nullspace), and calculate $A$’s rank. The row and column bases should consist only of rows/columns of the matrix $A$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For the columns:

1st column: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

2nd column: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

3rd column: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Free variables: $x_4$ and $x_5$

Rank = 3
2) (7 points) If $PB = LU$, find the general solution to the system $Bx = y$ where

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -6 & 1 & 2 \\ 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$Bx = y \implies PBx = Py$$

So solve $Lc = Py$

$Ux = c$

$Lc = Py = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \implies c = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$Ux = c \implies x = \begin{pmatrix} 4 \\ 10 \\ -1 \end{pmatrix}$

Set $\text{free} = 0$

$\text{free} = 1 \implies \text{null space} \implies \begin{pmatrix} \frac{7}{10} \\ 0 \end{pmatrix}$

So general solution is

$$\begin{pmatrix} 4 \\ 10 \\ -1 \end{pmatrix} + \frac{7}{10} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
3) a) (4 points) Vectors \( v_1 \) and \( v_2 \) are a basis for a subspace \( V \) of \( \mathbb{R}^5 \). If \( v_1 \cdot v_1 = 4 \), \( v_1 \cdot v_2 = -6 \), and \( v_2 \cdot v_2 = 10 \) use Gram-Schmidt to obtain an orthonormal basis for \( V \).

\[
v_2' = v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = v_2 + \frac{3}{2} v_1
\]

So normal \( \frac{v_1}{4} \), \( \frac{v_2 + \frac{3}{2} v_1}{\sqrt{100 - 18 + 36}} = \frac{v_2 + \frac{3}{2} v_1}{\sqrt{118}} \).

b) (3 points) Describe what you would do next if \( V \) was a subspace of larger dimension, and you had another vector \( v_3 \) in the basis.

Subtract projection of \( v_3 \) into \( v_1 \) or \( v_2 \). For \( v_3 \), then normalize it by dividing by its length.

c) (4 points) Describe the usefulness of the \( QR \) factorization of a matrix. How does it save time?

Since we can solve \( A^T A \tilde{x} = b \) by

\[
Q^T \tilde{x} = QTb
\]

back substitution to make it much faster!
4) (6 points) Consider the following inconsistent equations: \( x - y = -1 \), \( 2x + y = -1 \), \(-x + y = 2\), \( x - 2y = -1 \). Find the least squares best fit solution to these equations.

\[
\begin{bmatrix}
1 & 2 & -1 & 1

2 & -1 & 1 & -2

-1 & 1 & 2 & -1

1 & 1 & 1 & -2

\end{bmatrix}
\begin{bmatrix}
2

1

2

1

\end{bmatrix}
\]

\[
= \begin{bmatrix}
12 & -1

-1 & 2

2 & -1

1 & -2

\end{bmatrix}
\begin{bmatrix}
2

1

2

1

\end{bmatrix}
\]

So
\[
\begin{bmatrix}
7 & -2

-2 & 7

\end{bmatrix}
\begin{bmatrix}
y

p

\end{bmatrix}
= \begin{bmatrix}
-6

4

\end{bmatrix}
\]

\[
\hat{y} = \frac{1}{45} \begin{bmatrix}
7 & 2

-6 & 4

\end{bmatrix}
\begin{bmatrix}
-74/45

-16/45

\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{-4}{5}

\frac{2}{5}

\end{bmatrix}
\]

\[
(A^TA)^{-1}
\]

5) (a) (3 points) Define the term "basis." (Assume simpler concepts are already defined, such as "linear combination" and "subspace.")

(b) (3 points) A matrix has column space spanned by \((1, 1, 1, 0)^T\) and \((0, 1, 1, 1)^T\). Find a basis for its left nullspace.

a) A basis is a set of vectors which spans a subspace \(U\) which is linearly independent.

b) \[
\chi^T \begin{bmatrix}
1 & 0 & 0 & 0

0 & 1 & 0 & 0

0 & 0 & 1 & 0

0 & 0 & 0 & 1

\end{bmatrix} = 0
\Rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0

0 & 1 & 1 & 1

\end{bmatrix} \chi = 0
\]

\[
\chi = \begin{bmatrix}
-1

0

1

0

\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0

1

0

1

\end{bmatrix}
\]
7) (6 points) For each set below, explain why it is not a vector space over the real numbers.

a) The set of all matrices.

Different dimensions can't be added

b) The set of all $3 \times 3$ matrices with determinant zero.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) The set of all sequences of real numbers that contain no negative terms.

- $(\text{seq})$ contains negative

8) (7 points) Let $A$ and $B$ be $5 \times 5$ matrices, and let $\det(A) = 5$ and $\det(B) = 0$. Calculate each or explain why it's impossible:

a) $\det(A^T) = \det A = 5$

b) $\det(A^{-1}) = \frac{1}{\det A} < \frac{1}{5}$

c) $\det(AB) = \det A \det B = 0$

d) $\det(B^{-1}) = \frac{1}{\det B}$ DNE

e) $\det(A$ with its second and third row swapped) $\det = -\det A = -5$

f) $\det(A^2) = \det(A) \det(A) = 25$

g) $\det(A^T A) = \det A \det A = 25$
9) (10 points) Express the coupled differential equations \( \begin{cases} x' = x + 2y, \\ y' = 3x \end{cases} \), \( x(0) = 2, y(0) = 1 \) using matrices. Then use eigenvalues and eigenvectors to find an exact formula for \((x, y)\).

\[
\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
e^{12t} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

So \[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} e^{12t} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

10) (a) (3 points) The matrix \( A \) has the singular value decomposition \( A = U \Sigma V^T \).

Find an expression for \( (A^T) \), the pseudoinverse of \( A \).

\[
\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & U^T \end{pmatrix}
\]

Note: This is NOT the SVD, but \( \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) is an expression that works. To be SVD, it would need \( 1/2 \) and \( \pm 1 \) in the necessary order.

(b) (3 points) Find a singular value decomposition for \( A^T \).

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U^T \end{pmatrix}
\]
11) (4 points) Consider the matrix \[
\begin{pmatrix}
-3i & 4+2i & -6 & \sqrt{2} - i\sqrt{3} \\
-4+2i & 2i & 4i & 0 \\
6 & 4i & 0 & 2+i \\
-\sqrt{2} - i\sqrt{3} & 0 & -2+i & i
\end{pmatrix}
\]. What special kind of matrix is this (be as specific as possible). What does this tell you about its eigenvalues and eigenvectors? Again, be as specific as possible, without actually computing them!

It is shown - Hermitian \( \Rightarrow A^{\dagger} = -A \)

Hence, all its eigenvalues are pure imaginary and its eigenvectors are an orthogonal basis for \( \mathbb{C}^4 \).