1) (10 points) Find the $PA = LU$ factorization for the matrix $A$ below. Then find bases for the row space, column space, and nullspace of $A$ (don't bother with the left nullspace), and calculate $A$'s rank. The row and column bases should consist only of rows/columns of the matrix $A$.

$$A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}$$
2) (7 points) If \( PB = LU \), find the general solution to the system \( Bx = y \) where

\[
P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix},
U = \begin{pmatrix} 2 & -6 & 1 & 2 \\ 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix},
y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}
\]
3) a) (4 points) Vectors \( v_1 \) and \( v_2 \) are a basis for a subspace \( V \) of \( \mathbb{R}^5 \). If \( v_1 \cdot v_1 = 4 \), \( v_1 \cdot v_2 = -6 \), and \( v_2 \cdot v_2 = 10 \) use Gram-Schmidt to obtain an orthonormal basis for \( V \).

b) (3 points) Describe what you would do next if \( V \) was a subspace of larger dimension, and you had another vector \( v_3 \) in the basis.

c) (4 points) Describe the usefulness of the QR factorization of a matrix. How does it save time?
4) (6 points) Consider the following inconsistent equations: \( x - y = -1, \ 2x + y = -1, \ -x + y = 2, \ x - 2y = -1 \). Find the least squares best fit solution to these equations.

5) (a) (3 points) Define the term "basis." (Assume simpler concepts are already defined, such as "linear combination" and "subspace.")
(b) (3 points) A matrix has column space spanned by \((1, 1, 1, 0)^T\) and \((0, 1, 1, 1)^T\). Find a basis for its left nullspace.
7) (6 points) For each set below, explain why it is not a vector space over the real numbers.
   a) The set of all matrices.

   b) The set of all $3 \times 3$ matrices with determinant zero.

   c) The set of all sequences of real numbers that contain no negative terms.

8) (7 points) Let $A$ and $B$ be $5 \times 5$ matrices, and let $\det(A) = 5$ and $\det(B) = 0$. Calculate each or explain why it's impossible:
   a) $\det(A^T)$

   b) $\det(A^{-1})$

   c) $\det(AB)$

   d) $\det(B^T)$

   e) $\det(A$ with its second and third row swapped$)$

   f) $\det(A^2)$

   g) $\det(A^TA)$
9) (10 points) Express the coupled differential equations \[
\begin{cases}
x' = x + 2y \\
y' = 3x
\end{cases}
\] using matrices. Then use eigenvalues and eigenvectors to find an exact formula for \((x, y)\).

10) (a) (3 points) The matrix \(A\) has the singular value decomposition \(A = U \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T\).

Find an expression for \((A)^+\), the pseudoinverse of \(A\).

(b) (3 points) Find a singular value decomposition for \(A^T\).
11) (4 points) Consider the matrix

\[
\begin{pmatrix}
-3i & 4+2i & -6 & \sqrt{2} - i\sqrt{3} \\
-4+2i & 2i & 4i & 0 \\
6 & 4i & 0 & 2+i \\
-\sqrt{2} - i\sqrt{3} & 0 & -2+i & i
\end{pmatrix}
\]

kind of matrix is this (be as specific as possible). What does this tell you about its eigenvalues and eigenvectors? Again, be as specific as possible, without actually computing them!