Quick Review of Complex Numbers

Learning goals: students recall basic facts about complex numbers

Let’s review the basic facts of complex numbers.

- The complex numbers are expressions of the form \( a + bi \) with \( i^2 = -1 \)
- Adding is easy: \((a + bi) + (c + di) = (a + c) + (b + d)i\)
- Multiplying is distributive: \((a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i\)
- Complex conjugate: if \( z = a + bi \), then \( \bar{z} = a - bi \)
  - \( z + \bar{z} = 2a \) is always real; \( z - \bar{z} = 2bi \) is always purely imaginary
  - \( z \bar{z} = a^2 + b^2 \) is always nonnegative real, positive of \( z \neq 0 \), and is the square of the modulus of \( z \), \( |z| \)
  - Division can now be accomplished by \( z / w = z\bar{w} / w\bar{w} \), and since the denominator is real, it is easy to divide by it
  - Conjugation commutes with arithmetic: \( \overline{a + b} = \overline{a} + \overline{b}, \overline{ab} = \overline{a}\overline{b} \)
- Every complex number can be written in polar form, \( z = r \text{cis}(\theta) = r \cos(\theta) + ir \sin(\theta) \)
  - If \( z_1 = r_1 \text{cis}(\theta_1) \) and \( z_2 = r_2 \text{cis}(\theta_2) \), then \( z_1z_2 = r_1r_2 \text{cis}(\theta_1 + \theta_2) \) and \( \bar{z} = r \text{cis}(-\theta) \)
- It is often more to-the-point to write \( e^{i\theta} \) instead of \( \text{cis}(\theta) \)
- We can thus quickly take powers and roots: \( z^n = r^n e^{in\theta} \) and an \( n \)th root of \( z \) is \( r^{1/n} e^{i\theta/n} \)
- Especially important is the fact that \( e^{2\pi i} = 1 \)
- We can use this to find the \( n \)th roots of unity: if \( w = e^{2\pi i/n} \), then \( w, w^2, w^3, \ldots, w^n \) are the \( n \) \( n \)th roots of unity