Applications

Learning goals: students see some of the applications of symmetric matrices, such as positive definite matrices and rotation of axes.

There are many situations in mathematics and applied mathematics where matrices turn out to be symmetric, so the foregoing discussions aren’t just theoretical fluff. Let’s look at a few.

Pivots

One odd thing that happens for real symmetric nonsingular matrices, and that is that the eigenvalues and the pivots have the same signs. They may be vastly different from each other, but they have the same signs. (We assume no row swaps are needed, which is the usual case.)

Proof: Let $A = LDU = LDL^T$. Note: $L^T$ probably doesn’t equal $L^{-1}$ so this is not the diagonalization. Consider the function $L(t) = (1 - t)L + tl$. Notice the following facts: $L(t)$ is always lower triangular, with ones on the diagonal, so is never singular, and $L(0) = L$ while $L(1)$ is the identity. Now let $A(t) = L(t)DL(t)^T$. Again, $A(0) = A$ and $A(1) = D$. Also note that the pivots of $A(t)$ are the entries in $D$ for any $t$, and these don’t change. So the pivots of $A(0)$ and $A(1)$ are the same. Note also that the eigenvalues of $A(t)$ change in a continuous way from those of $A(0)$ to those of $A(1)$. Since the change is continuous, is any of these eigenvalues changed signs along the way, it would have to have gone through zero, making the determinant of that $A(t)$ zero which is impossible since all $A(t)$ are nonsingular. So no eigenvalue changes sign—as the move from those of $A(0) = A$ to those of $A(1) = D$ which is diagonal, so the eigenvalues are the pivots.

Quadratic forms

A nice application of symmetric matrices is to the theory of quadratic forms, which can be written $\sum_{i,j} c_{ij}x_ix_j$. That is, we take a bunch of variables, multiply them in pairs, multiply by a bunch of coefficients, and add them all together. If we let $a_{ii} = c_{ii}$ and $a_{ij} = \frac{c_{ij} + c_{ji}}{2}$, then we get the same values as outputs by using the quadratic form $\sum_{i,j} a_{ij}x_ix_j$. This is symmetric in $i$ and $j$. And it leads to the nifty matrix rendition $x^TAx$ where $x$ is the vector $(x_1, \ldots, x_n)$ and $A$ is the symmetric matrix of $a$’s. Thus, there is a one-to-one correspondence between symmetric matrices and quadratic forms.

One very important question about quadratic forms (and the corresponding symmetric matrix) is whether they always output positive numbers for non-zero $x$’s. This is the case, for example, for $x_1^2 + 2x_1x_2 + 3x_2^2$, which corresponds to the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. Such forms are called positive definite. They play an important role, for example, in the determination of the nature of
critical points of functions of several variables. The matrix of second partial derivatives (the Hessian matrix) is naturally symmetric, and if \( f(x) \) has a critical point (gradient = 0) and the Hessian is positive definite, the function has a minimum—just as in the second derivative test for functions of a single variable.

So how can we tell if a matrix is positive definite? Let’s look at \( A = SAS^{-1} = QΛQ^T \). Since \( A \) has a full set of eigenvectors, we know they form a basis for \( \mathbb{R}^n \). Let any eigenvalue of \( A \) be negative or zero. Then the corresponding eigenvector \( x \) has \( x^T A x = x^T λ x = λ(x^T x) \) which is negative or zero as \( λ \) is. So a matrix, or its corresponding quadratic form, is positive definite exactly when all of its eigenvalues are.

Now we have seen that if all the eigenvalues are positive, so must all the pivots be positive, since for symmetric matrices these sets of numbers have the same signs. And since the pivots are all positive, and they are revealed by elimination which does not alter the determinants of submatrices, all submatrices that are symmetrically located along the diagonal must have positive determinant.

The converse of this is also true. If all submatrices have positive determinant, then in particular the \( 1 \times 1, 2 \times 2, \ldots \) submatrices in the upper left corner must be positive. This will cause the first, second, … pivot to be positive, and so all the eigenvalues are positive.

**Rotation of axes**

When faced with the general quadratic equation \( Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0 \) (the 2 is there for convenience’ sake), latter completion of the square will rid ourselves of \( D \) and \( E \). This is essentially just shifting the coordinate axes—moving the origin to a more convenient location. We could also divide by \( F \) if not zero (if it is, we are faced with a degenerate conic, and by factoring \( Ax^2 + Bxy + Cy^2 = 0 \) we will get either a single point, a line, or two intersecting lines). Thus we are really faced with finding what happens with \( Ax^2 + Bxy + Cy^2 = 1 \). We are again looking at a quadratic form, with matrix \[
\begin{pmatrix}
A & B \\
B & C
\end{pmatrix}
\]. The important thing is to find its eigenvalues and eigenvectors. For eigenvalues of opposite signs give us a hyperbola, while the same sign makes for an ellipse (if one is zero, we get a parabola; both zero and the form isn’t quadratic!). The eigenvectors tell us the directions of the axes—the directions in which we should orient the axes to align them with the axes of the conic.

**Example:** Tell about the quadratic equation \( 2x^2 - 4xy - y^2 - 8x + 2y + 3 = 0 \).

**Solution:** we concentrate on the quadratic form \( 2x^2 - 4xy - y^2 \). The representative matrix is
\[
\begin{bmatrix}
2 & -2 \\
-2 & -1
\end{bmatrix}
\] (remember it’s 2\(B\) in there!)
\[
= \begin{pmatrix}
2/\sqrt{5} & 1/\sqrt{5} \\
-1/\sqrt{5} & 2/\sqrt{5}
\end{pmatrix}
\begin{pmatrix}
3 & 0 \\
0 & -2
\end{pmatrix}
\begin{pmatrix}
2/\sqrt{5} & -1/\sqrt{5} \\
1/\sqrt{5} & 2/\sqrt{5}
\end{pmatrix}.
\] So we should rotate our axes to be along the lines of slope 2 and slope \(-1/2\) (the “slopes” of the eigenvectors). We could do something like \( x = (x' + 2y')/\sqrt{5}, y = (2x' - y')/\sqrt{5} \). If we made this substitution into our equation, our equation would change to \(-2x'^2 + 3y'^2 + \) (lower degree stuff). From here we could easily complete the square and so forth to find the intimate information about the resulting hyperbola.

Reading 6.5

Problems: 6.5: 1 – 7, 8, 9, 10, 11, 12, 14,15, 17, 19, 20, 22, 26, 32