The Nullspace of a Matrix

Learning Goals: Students encounter another important subspace tied to a matrix, and learn how to find a set of vectors that describe it (a basis—even though we haven’t defined that yet!)

Another important subspace tied to a matrix is called its nullspace:

Definition: given an \( m \times n \) matrix \( A \), its nullspace \( \text{N}(A) \) is the set of all solutions to \( Ax = 0 \).

The nullspace is also called the kernel, especially if we are thinking of \( A \) as defining a function sending \( x \) to \( Ax \).

As opposed to the column space, which lives inside of \( \mathbb{R}^m \), the nullspace lives inside of \( \mathbb{R}^n \). It is indeed a subspace. Clearly it is non-empty, for \( x = 0 \) is certainly a solution to \( Ax = 0 \). If \( x \) and \( y \) are both solutions, then \( A(x + y) = Ax +Ay = 0 + 0 = 0 \), so the set is closed under addition. If \( k \) is a scalar, then \( A(kx) = k(Ax) = k0 = 0 \). So the set is also closed under scalar multiplication, and is thus a subspace.

Note that solutions to other right hand sides, \( Ax = b \), do not form a subspace. For one thing they don’t contain zero.

Example: consider the plane \( 2x - 3y + 7z = 0 \). This plane is a subspace because it passes through the origin. It is also the nullspace of the \( 1 \times 3 \) matrix \( A = \begin{bmatrix} 2 & -3 & 7 \end{bmatrix} \) because \( Ax = 0 \) multiplies out to give \( 2x - 3y + 7z = 0! \)

Example: find the nullspace of \( \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix} \). When we do a little elimination, it turns out that the second row vanishes entirely. That makes sense, because \( 9x + 3y = 0 \) is just a multiple of the first equation \( 3x + y = 0 \). So we really only have one equation. Thus, we can set \( y \) to be anything we want, and then solve for \( x \). The solutions are any \((x, y)\) of the form \((-t/3, t) = t(-1/3, 1)\), which is the line of slope \(-1/3\) through the origin.

Example: if \( A \) is invertible, then its nullspace is \( \mathbb{Z} \). This is because \( 0 \) is the only solution to the defining equation \( Ax = 0 \).

Notice how in the second example the nullspace was all multiples of a single special vector. This turns out to be the case in general. For a general \( m \times n \) matrix, we can find a set of “special” vectors so that the nullspace is the set of all linear combinations of these special vectors. In the first example, we can set \( y \) and \( z \) to be anything we want, and then solve for \( x \). That makes the solutions be all linear combinations of \((3/2, 1, 0)\) and \((-7/2, 0, 1)\). Why? In the equation \( 2x - 3y + 7z = 0 \) we may rewrite \( x = 3/2y - 7/2z \). Linear combinations of \((3/2, 1, 0)\) and \((-7/2, 0, 1)\) give \( y(3/2, 1, 0) + z(-7/2, 0, 1) = (3y/2 - 7z/2, y, z) \). The special thing about the vectors \((3/2, 1, 0)\) and \((-7/2, 0, 1)\) is that with the ones and zeroes in the \( y \) and \( z \) positions, we can always read off immediately what value we’ve chosen for \( y \) and \( z \).

Why do we choose \( y \) and \( z \) and solve for \( x \)? Why not choose \( x \) and \( z \) and solve for \( y \), for example? The reason is that the \( x \)-column has a pivot in it. Had there been other equations, we
would have had to solve for \( x \) in back substitution. So in isolation, we’ll just solve for \( x \) to keep things consistent.

This is the way it works in general.

**Example:** find the “special” vectors for the nullspace of
\[
\begin{bmatrix}
2 & 3 & -1 & 2 & 0 \\
-4 & -6 & -2 & -1 & 4
\end{bmatrix}
\]  

To start, we reduce to get
\[
\begin{bmatrix}
2 & 3 & -1 & 2 & 0 \\
0 & 0 & 0 & 3 & 4
\end{bmatrix}
\]  

The columns with pivots are the first and the fourth, so the variables \( x_1 \) and \( x_4 \) will be called *basic* or *pivot* variables. The variables \( x_2, x_3, \) and \( x_5 \) will be called *free* variables. Since we can set free variables to be anything we want, we will set each equal to one while setting the others to zero, and solve for the basic variables. This gives the vectors \((-3/2, 1, 0, 0, 0), (1/2, 0, 1, 0, 0)\) and \((4/3, 0, 0, -4/3, 1)\). The claim is that any “null vector” can be written as a combination of these three. The reason is that the null vector has to have some values for \( x_2, x_3, \) and \( x_5 \), and once these are known \( x_1 \) and \( x_4 \) are determined, and are the exact combinations of the free variables shown in these vectors.

We can get an even better view by continuing the elimination. First we will eliminate *upward* in each pivot column (producing
\[
\begin{bmatrix}
2 & 3 & -1 & 0 & -8/3 \\
0 & 0 & 0 & 3 & 4
\end{bmatrix}
\]  

and then dividing each row by its pivot (note that these operations don’t change the solution set!) giving
\[
R = \begin{bmatrix}
1 & 3/2 & -1/2 & 0 & -4/3 \\
0 & 0 & 0 & 1 & 4/3
\end{bmatrix}
\]  

From this matrix we can read off the special vectors simply by changing the sign of the non-pivot entries. Thus the special vector corresponding to \( x_5 \) is \((4/3, 0, 0, -4/3, 1)\). This form of the row-reduced matrix is called the *reduced row-echelon form* (RREF) of the matrix.