Vector Spaces and Subspaces

Learning Goals: students see what a vector space is, including several examples. Students encounter the idea of a subspace.

It is now time to move on to the theory surrounding linear equations. The key idea in solving a system $Ax = b$ is that we were looking for a column $x$ whose entries gave a linear combination of the columns of $A$ that added up to $b$. Instead of looking for a particular linear combination that works, let’s look at the problem in reverse. Let’s look at all the linear combinations of the columns of $A$ and see if our $b$ isn’t among them.

These collections of all linear combinations of things is the central concept of linear algebra. They are called vector spaces. Let’s look at a few.

The primary examples of vector spaces are $\mathbb{R}^n$ for various $n$. For instance, $\mathbb{R}^3$, the set of all column vectors of three entries is a vector space. So are $\mathbb{R}^2$ and $\mathbb{R}^5$, which just have different numbers of components. The number of components is called the dimension of the space. The key fact is that if we add any two such vectors, or multiply one by a constant, we get another such vector. In other words, we can take linear combinations of vectors in a space and get another vector in the space.

Even $\mathbb{R}^1$ counts as a vector space, which behaves just like the real numbers. And $\mathbb{R}^0$ is a vector space, containing just the zero vector. The book calls this $\mathbb{Z}$ (which should ideally be reserved to abbreviate the integers, but won’t cause any confusion for us because we will not need to refer to the integers in this course). Is the empty set a vector space? It is usually not allowed, because you can’t add things in it (there are no things to add!).

What are other instances where we can take linear combinations and get new things in the same set? Here are a few more vector spaces:

- $M_{2,2}$, the space of all $2 \times 2$ matrices of real numbers. We can certainly add them and multiply them by real numbers to get other members of $M_{2,2}$. You can probably guess that this vector space has 4 dimensions. There are similar vectors space $M_{m,n}$ of $m \times n$ matrices. Is the set of all matrices a vector space? No! Matrices of differing sizes can’t be added.

- $P_n$, the space of all polynomials of degree $n$ or less (including the 0 polynomial). These can also be added and multiplied by numbers (not by $x$, though!) to get new $n$th degree polynomials. The dimension is $n + 1$. Why? There are $n + 1$ choices of coefficients. Is the set of all polynomials a vector space? Yes! You can add them or multiply by constants and get polynomials.

- $\mathbb{C}^2$, the set of all columns of two complex numbers. $\mathbb{C}^n$ for that matter is a vector space. Actually, each is two different vector spaces, depending on whether you let the constant multiples be any complex number, or whether you force the multiples to be real numbers. Allowing the uses of different sets of constants changes the nature of the set’s properties.

For example, using all complex numbers, $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$ is a multiple of $\left[\begin{array}{c} i \\ -i \end{array}\right]$ (namely, it is $-i$ times as much). But using only real numbers, there is no multiple of the one that will give you the other. We could get even weirder by allowing the constants to be, say, only...
rational numbers. In this course, we will almost always consider complex vectors with complex constants, and real vectors with real constants.

- \( C^1 \), the set of all continuously differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \). We can add two functions and multiply them by constants and they remain differentiable. This vector space has an infinite number of dimensions.

There are actually ten rules that a set has to follow in order to be a vector space. Here they are (these are not in the same order as the text has them):

If \( u, v, \) and \( w \) are in the set and \( k \) and \( l \) are constants (formally called scalars) then

1. \( u + v \) is in the set (additive closure)
2. \( ku \) is in the set (scalar multiplicative closure)
3. \( u + v = v + u \) (commutativity)
4. \((u + v) + w = u + (v + w)\) (associativity)
5. There is a unique element \( 0 \) such that \( x + 0 = x \) for all elements of the set (additive identity)
6. For each \( x \) there is a \(-x\) such that \( x + (-x) = 0 \) (additive inverse)
7. \( 1x = x \) for all \( x \) (scalar multiplicative identity)
8. \( k(lu) = (kl)u \) (scalar multiplicative associativity)
9. \( k(u + v) = ku + kv \) (vector distributivity)
10. \( (k + l)u = ku + lu \) (scalar distributivity)

For many sets, most of these rules are more-or-less automatically true. They are direct results of the way in which the operations are defined on the set. For instance, the truth of these statements for \( \mathbb{M}_{2,2} \) depends on the truth of these same facts for real numbers and a little bookkeeping to make sure all the results go into the right row and column.

From these, we could prove a bunch of corollaries. For instance, \( 0x = 0 \) for any \( x \). This is because \( kx = (k + 0)x = kx + 0x \), so \( 0x \) acts just like \( 0 \), and since \( 0 \) is unique \( 0x \) must be \( 0 \). We will assume all these mundane consequences of the axioms proved without further comment.

**Subspaces**

Many times one vector space will live inside another. For instance, the space of 3\(^{rd}\) degree polynomials lives inside the space of all polynomials. In \( \mathbb{R}^3 \), any plane through the origin and any line through the origin are vector spaces in their own right, living inside the larger vector space. Actually, the origin all by itself, \( \{0\} \) is a vector space, too!

**Definition:** If \( V \) is a vector space and \( W \) is a subset of \( V \) that is a vector space in its own right using the arithmetic rules from \( V \), then \( W \) is called a subspace of \( V \).

It is important the \( W \) use the same arithmetic rules as \( V \), otherwise it’s just too confusing to keep track of what’s going on. Besides, if \( W \) uses all the same arithmetic rules as \( V \), we don’t need to check all ten axioms above.

The reason is that the truth of most of those axioms will be inherited from the parent space to the daughter space. For instance, commutativity and associativity are true if we restrict ourselves to any subset of \( V \). Which ones do we still have to check? The two defining qualities of vector spaces! The two closure rules are all that need to be checked.
Theorem: If a non-empty subset of vector space $V$ is closed under addition and scalar multiplication, it is a subspace of $V$.

Proof: the only other axioms that aren’t automatically inherited by the subset are the existence of zero and of inverses. But take any $v$ in the subspace, and multiply it by the scalar 0. $0v = 0$, so by multiplicative closure 0 is in the subset. Similarly, multiplying by the scalar $-1$ puts $-v$ in the subset.

Example: the set of symmetric $2 \times 2$ matrices is a subspace of $M_{2,2}$. For adding symmetric matrices and multiplying them by scalars leaves symmetric matrices.

Example: the first quadrant in $\mathbb{R}^2$ is not a subspace. It is closed under addition, but not multiplication (by negatives).

Example: if we throw in the third quadrant to fix multiplicative closure, we mess up additive closure, for $(2, 1) + (-1, -2) = (1, -1)$ which is in the fourth quadrant.

Example: the set of continuously differentiable function with $f(3) = 0$ is a subspace, because adding two such or multiplying one by a constant leaves a function with a zero at $x = 3$.

Example: The subspaces of $\mathbb{R}^3$ are planes through the origin, lines through the origin, the zero vector, and the entire space $\mathbb{R}^3$.

What are scalars?

What do we allow as scalars? The real numbers must certainly be allowable. What do we want to do with them? We need to be able to add and multiply them, basically to be able to do any kind of normal arithmetic. The set of scalars needs to be a field. It has to have an addition and a multiplication that satisfy the following rules:

- Addition and multiplication are associative
- Addition and multiplication are commutative
- Addition and multiplication have identities
- Addition has inverses; multiplication has inverses except for 0
- Multiplication distributes over addition

There are concepts similar to vector spaces where we allow more general sets of scalars (such as the integers, with no inverses) called modules, and most of the things about vector spaces remain true, but are much harder to prove!