$A = LU$ and Solving Systems

Learning goal: to solve a general system $A\mathbf{x} = \mathbf{b}$ if given the $A = LU$ factorization

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You may say to yourself, "that $A = LU$ business is sort of interesting, but can it help me solve systems?" The answer is a resounding YES. It is, in fact, exactly the same efficiency as doing Gaussian elimination (it *IS* Gaussian elimination) if we solve a single system. The real benefit comes if we have more than one system to solve.

So let $A = LU$ and assume our job is to solve $A\mathbf{x} = \mathbf{b}$. That is, we need to solve $LU\mathbf{x} = \mathbf{b}$. Since we reduced with no hitches, there is some $\mathbf{x}$ that solves this system. Let $U\mathbf{x} = \mathbf{c}$. In other words, we are *defining* $\mathbf{c}$ to be $U\mathbf{x}$.

Then we have that $L\mathbf{c} = \mathbf{b}$. We don't know what $\mathbf{c}$ is, because we don't know what $\mathbf{x}$ is. But we can *solve* for $\mathbf{c}$ by solving $L\mathbf{c} = \mathbf{b}$. Note that since L is lower triangular, this is a really easy system to solve, because it can be done by forward substitution—the same idea as back substitution, only you start from the first variable and work your way forward. So we can find $\mathbf{c}$ very efficiently.

Another way to look at this is the $\mathbf{c} = L^{-1}\mathbf{b}$. L^{-1} exists because it is simply the product of the matrices that performed elimination on A. Thought of this way, $\mathbf{c}$ is what we get from $\mathbf{b}$ by performing the steps of elimination on it—which is what we have to do in Gaussian elimination anyway!

Since $\mathbf{c}$ is the right-hand side we would get from performing elimination on $A\mathbf{x} = \mathbf{b}$ to get to triangular form anyway, we finish up by doing back substitution on $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$.

Theorem: to solve $A\mathbf{x} = \mathbf{b}$, if $A = LU$ it is equivalent to solve $L\mathbf{c} = \mathbf{b}$ and then $U\mathbf{x} = \mathbf{c}$.

The real benefit comes when we decide to solve another linear system with the same left-hand side but a different $\mathbf{b}$. Since the elimination has already been done, we save a lot of work by simply solving the pair of triangular systems for the new $\mathbf{b}$!

As an example, let's solve $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -4 & -4 & 3 & -2 \\ 2 & 9 & 1 & 4 \\ 4 & 6 & -1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ -11 \\ 17 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -4 \\ 15 \\ 8 \end{bmatrix}$ as two different right hand sides. Now $A = LU$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

First we solve $L\mathbf{c} = \mathbf{b}$ for the first right hand side: $c_1 = 6$; then $-2\cdot 6 + c_2 = -11$, so $c_2 = 1$; then $1\cdot 6 + 3\cdot 1 + c_3 = 17$, so $c_3 = 8$, and finally $2\cdot 6 - 1\cdot 8 + c_4 = 16$, so $c_4 = 12$. It is worth checking that if we had done elimination on the original system $A\mathbf{x} = \mathbf{b}$, this is exactly the right hand side we would have obtained. Now solve $U\mathbf{x} = \mathbf{c}$. This time we back substitute: $4x_4 = 12$, so $x_4 = 3$. Then $-1\cdot x_3 + 3\cdot 3 = 8$, so $x_3 = 1$. Next $2\cdot x_2 + 1\cdot 1 = 1$, so $x_2 = 0$. Finally $2x_1 + 3\cdot 0 - 1\cdot 1 + 1\cdot 3 = 6$, so $x_1 = 2$. The solution is (2, 0, 1, 3) which is easily checked.

We can then similarly solve the system for the second right-hand side. Again, elimination is already done, so all we have to do is forward substitution, then back substitution. We quickly find **c** = (3, 2, 6, 8) and **x** = (–1, 1, 0, 2), which can also be easily checked.

Side note

Though it has nothing to do with solving systems, we might note here that $A = LU$ isn't "symmetric" in the sense that L has ones on the main diagonal, but U does not. Sometimes we modify U by pulling the pivot out of each row. Recall that dividing a row by a non-zero entry is an elementary operation. To do this, we multiply U on the left by a matrix *on the left* to multiply each *row* by a number. That is, we replace LU with LDU, where D is the diagonal matrix of pivots, and this new U is the old U with each row divided by its pivot. For instance, in our 4×4 example, $U = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is replaced by $DU = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3/2 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. It is unfortunate, but we call this new U by the same name as the old U!

The really nice thing about $A = LDU$ factorization is that it is *unique!* Just like factoring integers is unique, we have a unique way of factoring a matrix into LDU form. The proof is simple:

Theorem: If A can be row-reduced without row swaps, then A has a unique factorization $A = LDU$ where L is lower triangular with ones on the diagonal, U is upper triangular with ones on the diagonal, and D is diagonal.

> **Proof:** Let LDU and $L'D'U'$ be two such factorizations. So $LDU = L'D'U'$. Multiplying by inverses in the appropriate places gives us $L'^{-1}L = D'U'U^{-1}D^{-1}$ (U and D are invertible because each is already row-reduced!). Note that the left side of this is lower triangular with ones on the diagonal, so the right side must be also. But the right side is upper triangular! The only matrix that is both lower- and upper- triangular is diagonal, so the left side is the diagonal matrix with ones on the diagonal—the identity!. Thus $L = L'$. Then we can put the D's on one side and the U's on the other, and both now must be I, so the U's are equal, as are the D's.

As a final note, we could also have pulled the pivots out of each *column* instead of the row, by writing LUD instead. This new form isn't as symmetric as the LDU form, although it is also unique (the L's and D's are the same as before, but the U is different).