Consider $\sum a_n$ with $a_n \geq 0$.

Then { S_n } is a non-decreasing sequence. What does that mean?

If we consider this sequence { S_n }, where we have

$$S_1 \leq S_2 \leq S_3 \leq . . . . \leq S_n \leq S_{n+1} \leq . . . .,$$

either this sequence will increase without bound OR it will be bounded above.

If we can find a number M such that if $S_n \leq M$ for all n, then we call M an upper bound of the sequence.

If a sequence is non-decreasing and it is bounded above, then the sequence will converge.

Ex: Let $S_n = \dfrac{2n}{3n+1}$.

Find an upper bound of the sequence.

Find three more upper bounds of the sequence.

Which of these do you think is the most significant?

Why? In other words, what makes it special?

Theorem: Let $\{S_n\}$ be a non-decreasing sequence.

Either (1) If an upper bound exists, then there is a Least Upper Bound L and the sequence converges to L.

Or (2) The sequence diverges to $+\infty$. (This means the S_n eventually exceeds every given finite M.)

Consider the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. While the a_n approach 0, it is not as clear as to whether the series converges or diverges. Graph the terms of the series and the S_n. You may wish to use the Series program with a few different values of n to see if it seems to converge. Any guesses about the convergence?

Theorem: The harmonic series diverges.

Proof: We need to show that the sequence of partial sums $\{ S_n \}$ increases without bound. This can be done by showing that S_n can be made arbitrarily large by taking an appropriate value of n – a value which is sufficiently large .

Consider the following:

$$S_2 = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$S_4 = S_2 + \_\_\_\_\_ + \_\_\_\_\_ > S_2 + \_\_\_\_\_\_\_ + \_\_\_\_\_ = S_2 + \frac{1}{2} > \frac{3}{2}$$

$$S_8 = S_4 + \_\_\_\_\_ + \_\_\_\_ + \_\_\_\_\_ + \_\_\_\_ > S_4 + \_\_\_\_\_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

$$= S_4 + \frac{1}{2} > \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

$$S_{16} = \ldots\ldots\ldots > \ldots\ldots\ldots = S_8 + \_\_\_\_\_\_\_ > \_\_\_\_\_\_\_\_\_\_$$

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$S_{2n} > \_\_\_\_\_\_$

Since this last expression (written in the blank above) is unbounded as $n \to \infty$, and since S_{2n} will always be larger, the sequence of partial sums increases without bound. Hence, the harmonic series diverges.