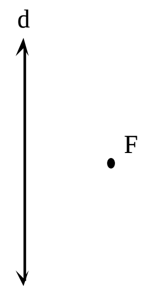
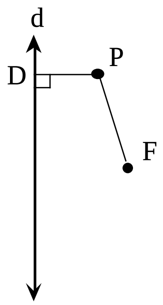
VARIABLE ECCENTRICITIES

by: Charles L. Hamberg and Susan K. Eddins
Illinois Mathematics and Science Academy

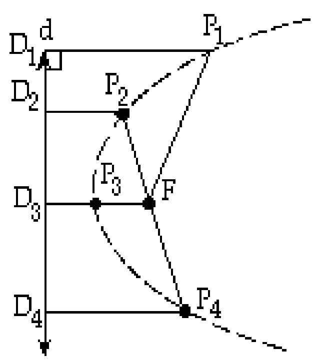
One approach used in defining and determining a conic begins with two simple geometric objects: a fixed line called a *directrix*, and a fixed point called a *focus*.



Suppose that we are given, a fixed line, d, and a fixed point, F. We can consider a conic to be the set (locus) of all points, P, on the plane where the ratio of its distance from the point F (PF) and its distance from the Line d (PD) is a constant.



That is, $\frac{FP}{PD} = e$ where e is a constant called the *eccentricity* of the conic.

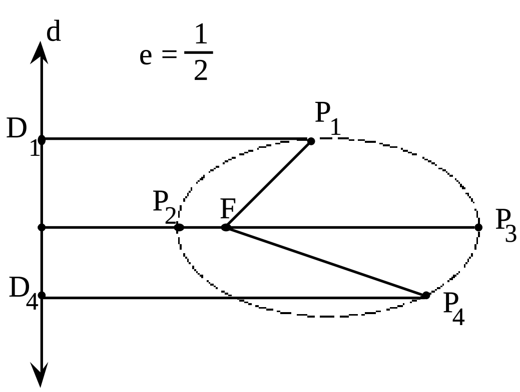


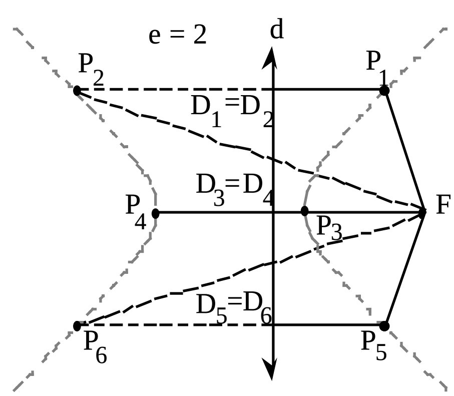
If we fix e to be equal to 1, we have a parabola.

Thus, if $\frac{FP_1}{P_1D_1} = \frac{FP_2}{P_2D_2} = \frac{FP_3}{P_3D_3} = \frac{FP_4}{P_4D_4} = 1$
,

then P_1, P_2, P_3, and P_4 are points on a parabola.

We can change our conic by changing the fixed value of the eccentricity. If the value of the ratios is fixed as being less than 1 the points lie on an ellipse. If the value of e is fixed as being greater than 1 the points lie on a hyperbola.

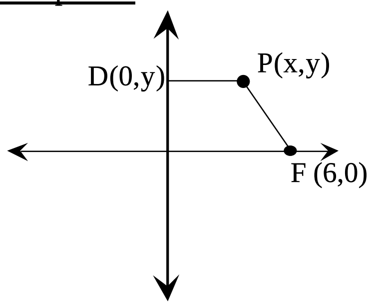




We note that in all previous examples the ratio **e** is fixed; a constant function. Suppose we were to let the value of **e** vary according to some equation; i.e. become a function of x. What types of curves might be produced? This question was first posed by a student, Mark Moody, at Adlai E. Stevenson High School, in the Fall of 1969. With the availability of the graphing calculator his question is more approachable today than it was then.

Let us consider some specific examples. In each Let $x = 0$ be a directrix and let F(6,0) be the focus.

<u>Example #1</u>:



Suppose we locate all points P(x,y) so that, $e(x) = x$ for all $x > 0$ that is $\frac{FP}{PD} = x$ for $x > 0$.

Thus, we have

$$\frac{FP}{PD} = \frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = x$$

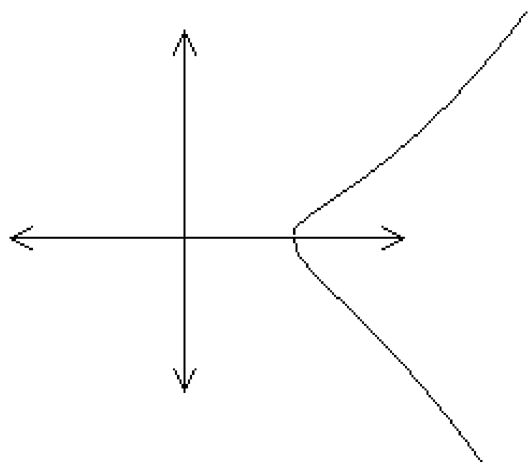
Squaring and rewriting gives us

$$y^2 = x^4 - (x-6)^2$$

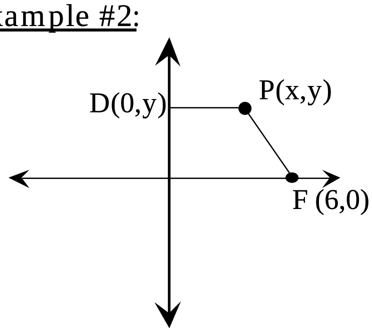
or

$$y = \pm\sqrt{x^4 - (x-6)^2}$$

1. A graphing calculator shows this graph of the relation for $x > 0$. For what domain is this relation defined?
2. What is the range?
3. What shape does the curve seem to approximate?
4. What would the graph be if the restriction $x > 0$ were removed?



<u>Example #2</u>:



Now consider $e(x) = \frac{FP}{PD} = x^2$ for $x > 0$.

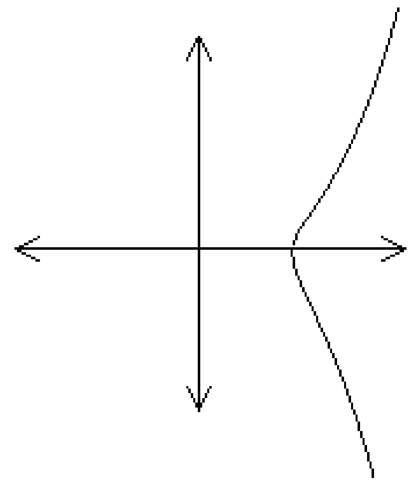
$$\frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = x^2$$

then, $y^2 = x^6 - (x-6)^2$

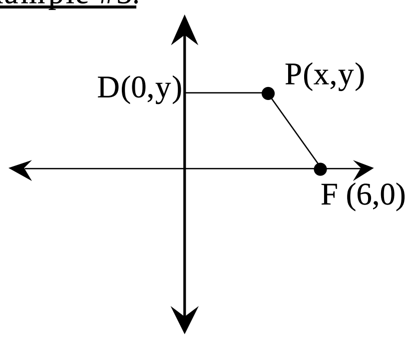
and

$$y = \pm\sqrt{x^6 - (x-6)^2}$$

1. Compare this graph and its domain with that found in Example #1.



Example #3:



Now look carefully at what happens when $e(x) = \frac{FP}{PD} = \frac{1}{x}$.

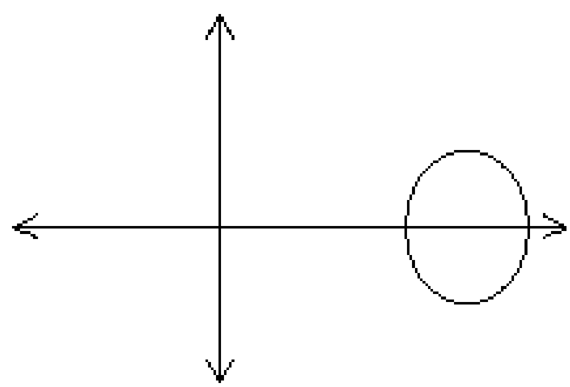
$$\frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = \frac{1}{x}$$

Squaring and simplifying we get

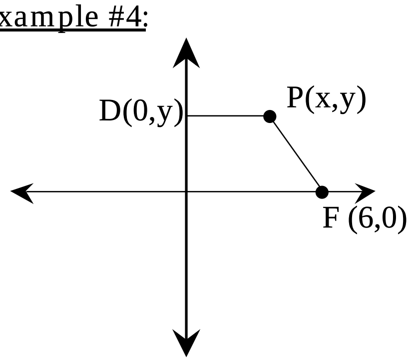
$$y^2 = 1 - (x-6)^2 \quad \text{(a)}$$

$$y = \pm\sqrt{1-(x-6)^2} \quad \text{(b)}$$

1. What is the domain for this relation when $x > 0$?
2. Consider equation (a). What relation does this equation describe?
3. Graph the relation with equation (b) on your graphing calculator using a window $\begin{matrix} 3 \le x \le 9 \\ -2 \le y \le 2 \end{matrix}$



Example #4:



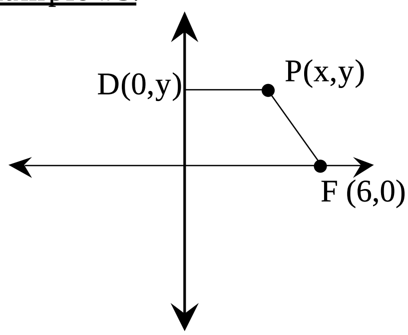
Repeat this process for $e(x) = \frac{FP}{PD} = \frac{1}{x^2}$.

See if you can "find" the graph!

1. Determine the domain for this relation.

Finally, consider a very different kind of eccentricity function.

Example #5:



What would the graph be if

$$e(x) = \frac{FP}{PD} = |\sin x| \quad \text{for } x > 0?$$

$$\frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = |\sin x|$$

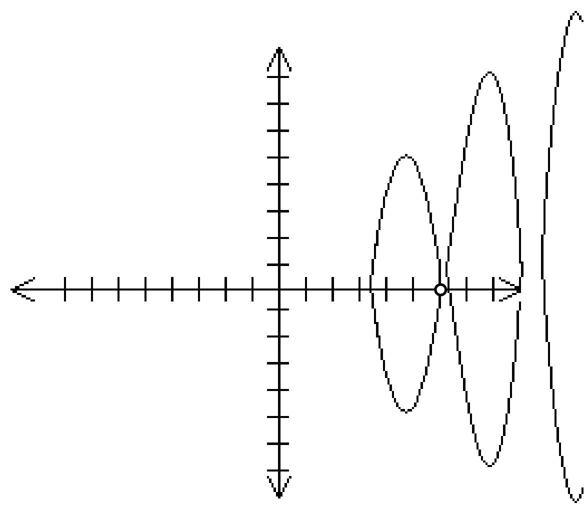
$$y^2 = x^2 \sin^2 x - (x-6)^2$$

or

$$y = \pm\sqrt{x^2 \sin^2 x - (x-6)^2}$$

Questions for investigation:

1. The graph of the relation for $x > 0$ is shown at the right. For what positive domain is the relation defined?
2. What is the range?
3. What shape, if any, does the curve seem to approximate?
4. Replace the ratio $|\sin x|$ by $\sin x$. How does that affect the graph?



If eccentricities are allowed to be functions of x then the resulting relations to be graphed require the use of technology. There is much here for students to explore and investigate.

The interested reader might try to explore what happens to the graph as x continues to increase. ✍