PROBLEM SETS à la IMSA

by: the Mathematics Faculty
Illinois Mathematics and Science Academy

The mathematics faculty of the Illinois Mathematics and Science Academy has developed a three semester sequence of courses, Mathematical Investigations I, II, and III, whose intent is to take students with a background in elementary algebra and geometry and help them prepare for the study of calculus. An integral part of these courses are sets of 30 - 40 problems given to students approximately once every six to seven school days.

Each *Problem Set* attempts to serve several purposes. It provides a review of material we *expect* that students have seen before but might need to bring to *present memory*, it contains problems which reinforce some of the current material, it previews constructs which will be useful in future work in mathematics, and it strives to present problems in a variety of contexts many of which link several ideas together in perhaps innovative ways - though this may be a subjective judgment.

Many of the articles which appear in the IMSA Math Journal have been taken from ideas developed for the Problem Sets. In this issue we have decided to include one Problem Set taken from the fall, 1995, Mathematical Investigations I course in its entirety. We hope you find it interesting. We would appreciate your feedback.

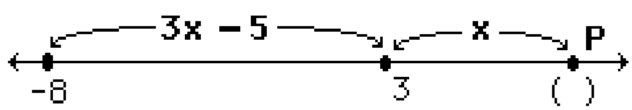
"MATHEMATICAL INVESTIGATIONS"-1

Problem Set - 4

1. The January price of a complete set of Bubs Baseball Cards went up by 20% in February. In March, the set went on sale at 20% off. By what percent did the price of a set of cards change from the January price ? (Specify increase / decrease)

2. Solve for **n**: $\dfrac{(a^n a^3)^2}{(a^6)^3} = a^{13}$

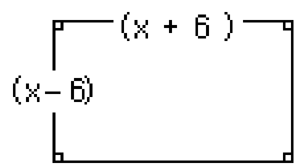
3. Find the coordinate of the point P, to the right of 3, on the number line shown.



4. Solve over the complex numbers: $2x + \frac{3}{2} = x^2$.

5. Find all integer values of x for which $\dfrac{6}{x+2}$ is an integer.

6. Find all values of **x** for which the area of the rectangle is less than 64. Write your answer in the form: $a < x < b$



7. Graph the set of ordered pairs, (**x**,**y**), for which **x** = 6 + **t** and **y** = 8 – 2**t** for $-4 \le t \le 4$ [Label the endpoints.]

8. Solve the system: $\begin{cases} 180x - 120y = 420 \\ \frac{x}{7} + \frac{y}{14} = \frac{1}{2} \end{cases}$ for (x,y)

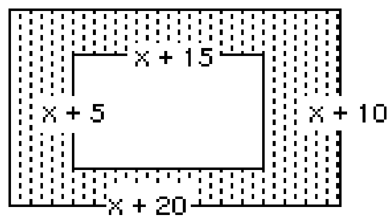
9. Recall: $n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

   Find k: a) $\frac{122!}{119!} = k$

   b) $93! + 92! + 91! = k^2 \cdot 91!$

10. Solve for x: $x^{\frac{2}{3}} - 5 = 59$

11. Determine x, if the shaded area between the two rectangles is 875 square units.



12. Let **a** $*$ **b** = ax + by for all real numbers **a** and **b.**
Find (x,y), if 2 $*$ 7 = 36 and (–2) $*$ 4 = 41.

13. P varies directly as the <u>square</u> of Q and Q = 10 when P = 6.
Find the value of P when Q = 8.

14. Solve the system: $\begin{cases} x^2 - y^2 = 63 \\ x - y = 7 \end{cases}$

15. □ QUIK is transformed to □ Q′U′I′K′ by 3 transformations.
□ QUIK has vertices Q(8,2), U(5,4), I (0,0), K(6,–1) □ Q′U′I′K′ has vertices Q′(11,–4), U′ (8,–6), I′ (3,–2), K′ (9,–1)

   a) Describe geometrically the transformations which have taken place.
   b) Write the matrix equation(s) to describe the transformation action.

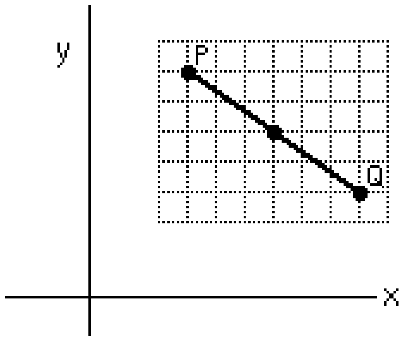
16. $C(n,r) = \frac{n!}{r!(n-r)!}$ Find C(92,88)

17. x is strictly within 5 units of –3.

a) Graph this set of values on a number line.

b) Write this set in set builder notation: Use the form: $\{x \mid |x - h| < r\}$

18.



Find the slope of $\overline{PQ}$.
[Careful! Scale is not given from origin.]

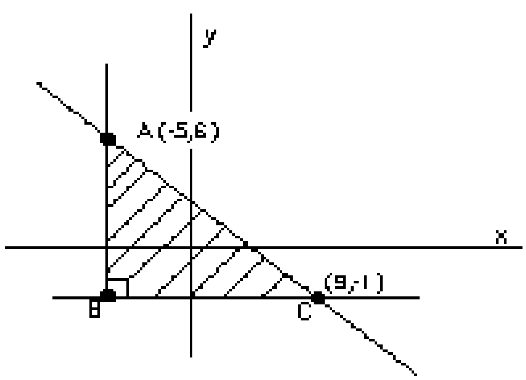
19. For demographic purposes, the U.S. Census Bureau classifies all areas of the country as being either rural, urban, or suburban. In a recent census, it was found that 27% of all adult U.S. citizens had, at some time in their lives, lived in a rural area; 74% had lived in an urban area; and 47% had lived in a suburban area. 7% of the adults had, at some time in their lives, lived in both rural and suburban areas; 27% had lived in both urban and suburban areas; and 16% had lived in both rural and urban areas.

a) What percentage of adult Americans have spent their entire lives in one single type of area (rural, urban or suburban)?

b) What percentage of adult Americans have lived in all three types of areas?

20. Find a <u>cubic</u> equation whose solutions are: $x = 0$, $x = -4$ and $x = 6$

21.



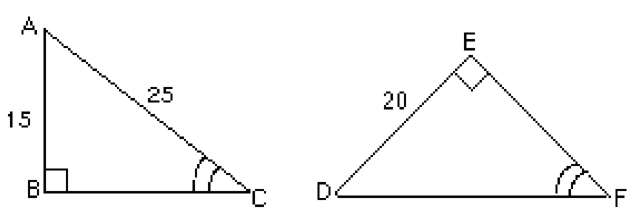
Find the equation of:

a) $\overline{AB}$

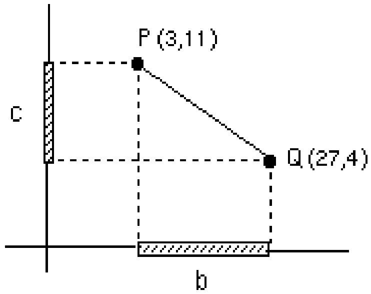
b) $\overline{BC}$

c) $\overline{AC}$

22.



Find the perimeter of ΔDEF.

23. 

a) Find the slope of $\overline{PQ}$

b) Find the shadow of $\overline{PQ}$ on the x axis. Write as a set, $\{x \mid ? \le x \le ?\}$

c) Find the shadow of $\overline{PQ}$ on the y axis. Write as a set, $\{y \mid ? \le y \le ?\}$

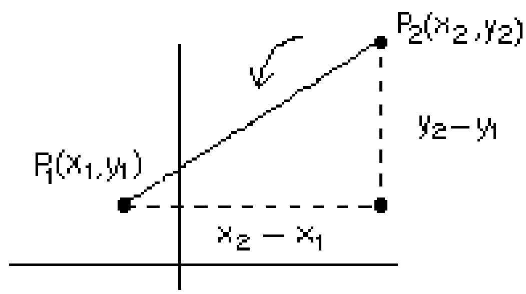
24. Define: a ® b = arithmetic mean between a and b.
a ¥ b = geometric mean between a and b.

a) Find 13 ® (9 ¥ 4) b) Find (19 ® 13) ¥ 4

c) If $a^2 + b^2 = 146$ and a ® b = 7, find a ¥ b.

25. The distance between two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

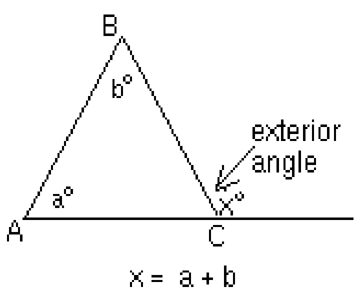


Find the perimeter of ΔFGH to the
F = (–12,–3) G = (4,7) H = (9,17)

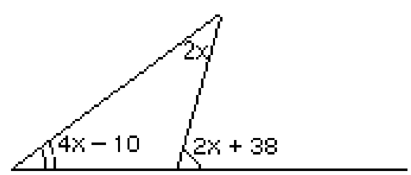
26. The vertex matrix of ΔABC is $\begin{bmatrix} 4 & a & 10 \\ -3 & a+5 & 2 \end{bmatrix}$. Determine **a** if the area of ΔABC is 100.

27. Simplify: $\dfrac{8}{\sqrt{2}} - 3\sqrt{8} + \frac{2}{3}\sqrt{288}$

28. An exterior angle of a triangle is equal to the sum of the remote interior angles.



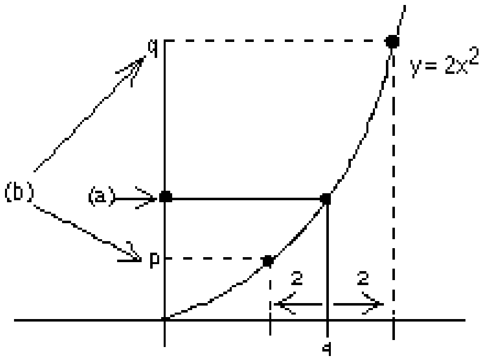
a) Solve for **x**.



b) The angles of a triangle are in the ratio of

29. $\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$

30.



a) Find y, if x = 4

b) Find p and q

31. Simplifying Radicals.

a) Multiply and simplify $(\sqrt{5} - 2)(\sqrt{5} + 2)$

b) Multiply and simplify $\dfrac{6}{(\sqrt{5} - 2)} \cdot \dfrac{(\sqrt{5} + 2)}{(\sqrt{5} + 2)}$

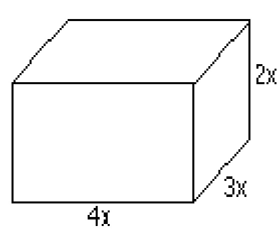
c) Rationalize denominator: $\dfrac{18}{\sqrt{7} - 4}$

32. $A = \begin{bmatrix} 5 & -3 \\ 0 & -1 \\ 6 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 4 & -3 \\ 2 & 0 & 1 \end{bmatrix}$ Find A • B and B • A

33. If $x + \dfrac{1}{x} = 5$, find $x^2 + \dfrac{1}{x^2}$

34. The edges of a rectangular box are in a ratio: length : width : height = 4 : 3 : 2

The perimeter of the smallest face



35. Given 4 rays **(no two opposite)** with a common endpoint, 6 angles are determined.

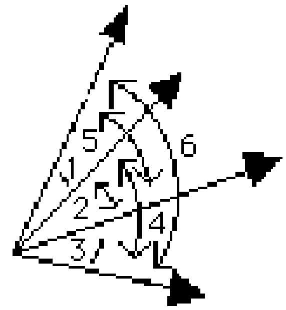
How many angles are formed by:

a) 3 rays with a common endpoint.

b) 5 rays with a common endpoint

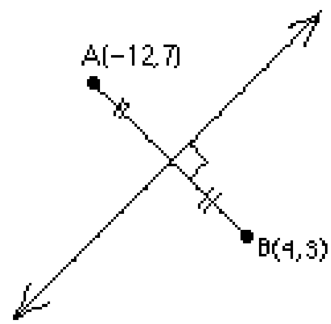
c) 6 rays with a common endpoint

d) n rays with a common endpoint



36. Solve for **x**: $\begin{vmatrix} x^2 & -6 \\ -4 & 3 \end{vmatrix} = 4 - 5x$

37. Find the equation of the line that is



38. a) Graph the line: $x - 3y = 7$

b) Find 2 ordered pairs that are integral solutions to the equation.

c) Shade the region where $x - 3y \leq 7$.

39. If Venus has a diameter of 12,100 km. while Mars has a circumference of 21,330 km., which is the **LARGER** planet? Justify.

40. Simplify:

a) $\dfrac{4! - 3!}{3!}$ b) $\dfrac{10! - 9!}{9!}$ c) $\dfrac{n! - (n-1)!}{(n-1)!}$

A commentary on selected problems:

- Problem #5 has variations which appear in other problem sets. Students are asked to find, for example, all *natural numbers* x for which the quotient is an integer or find all integer values of x for which the quotient is a *positive integer*.

- In problem #6 and again in #11 restrictions on the domain come from the fact that each side measurement as well as the area must be positive while the area is also less than 64. Students get pretty good after awhile in looking for *natural* domain restrictions in a variety of settings. Visualization enhances this process.

- Problem #9 gives students an "operational definition" for *factorial.* The goal is to increase students' reading ability in math. As a by-product, some topics are not formally "taught" in the classroom. They are introduced, defined, and practiced solely in problem sets. In addition, #9b leads to an interesting conjecture.

- We have some "favorite problem" types which are often included in problem sets. Problem #24c is a "sum-and-product" problem. Such problems give the student information on two out of three pieces of information (the sum of two numbers, their product, and the sum of their squares) and they are asked to find the missing information. An "elegant" or "clever" way to approach the problem begins by noticing that squaring the binomial relates these three expressions.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

In #24: $\left(\frac{a + b}{2}\right) = 7$ so $(a + b) = 14$ and we have $a^2 + b^2 = 146$

$$196 = 146 + 2ab$$

$$ab = 25$$

$$a¥b = \pm 5$$

Later in the year students love to look for sum-and-product problems and identify all the ways we have "disguised" them in a problem set. Problem #14 looks like it might be a sum-and-product problem, but it isn't. Problem #33 is!

- Problems #23 and #30 "preview" visual constructs from calculus. Approached from the idea of "shadows" or intervals on the axes groundwork for the idea of a neighborhood can be made easily accessible. Problem #23 also relates well to later work in math or physics involving horizontal and vertical components of vectors.

- We began the year with work on geometric transformations described by matrix operations. To review those operations and see matrix operations in a new setting, they are used to present a system of equations in Problem # 29.