SOME EXPLORATIONS WITH EVEN AND ODD FUNCTIONS

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Mathematics textbooks often include a brief discussion of odd and even functions, particularly as they relate to symmetries of graphs about the x-axis and the origin. In this article we shall consider several exercises which deal with odd and even functions not only with regard to their graphs but also by asking students to carefully use definitions, form conjectures, give convincing counter examples, or validate their conclusions. In our work we will limit our discussion to odd and even functions of x.

Exercises

Definition: A function f is even if $f(x) = f(-x)$ for all values of x in the domain of f.

1. Explain why the following functions are even functions:

   (a) $f(x) = |x|$ (b) $g(x) = \cos x$

   (c) $h(x) = 2^x + 2^{-x}$ (d) $j(x) = \begin{cases} y = t^6 \\ x = t^3 \end{cases}$

   (e) $k(x) = 6$ (f) $l(x) = x^4 - x^2 + 3$

2. Explain why a function f is even if $f(x) - f(-x) = 0$ for all values of x in the domain of f.

3. Explain why $f(x) = x^2$ is not an even function for x $[-2,4]$.

4. Explain why $f(x) = \dfrac{x^4 - x^2 + 6}{(x-3)(x+1)}$ is not an even function.

Definition: A function f is an odd function if $f(x) = -f(-x)$ for all values of x in the domain of f. Alternatively, if $f(x) + f(-x) = 0$ for all values of x in the domain of f.

5. Describe which of the following functions are even functions, which are odd functions and which are neither.

   (a) $f(x) = x^3$ (b) $g(x) = \log|x|$ c) $h(x) = 2^x - 2^{-x}$

   (d) $j(x) = \begin{cases} y = t^9 \\ x = t^3 \end{cases}$ (e) $k(x) = x^7 - x^3 + x$ f) $l(x) = (x+3)^2$

6. Explain why $y = \frac{x(x-2)(x+2)}{(x+1)(x-1)}$ is an odd function.

7. In the study of calculus, we will see that,
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \ldots$ for all x
and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \ldots$. Using these expressions, explain why cos x is even and sin x is odd.

Some Further Exercises

1. Explain why the sum of two even functions is an even function and the sum of two odd fuctions is an odd function.

2. (a) Given an example to show that the sum of 2 even functions can be **both** even and odd.

   (b) Give an example to show that the sum of 2 odd functions can be **both** even and odd.

3. Explain why the product of two odd functions will **always** be an even function.

4. (a) $(a,f(a))$, $a \neq 0$, is on the graph of even function $y = f(x)$. What other point is guaranteed to be on the graph?

   (b) $(a,g(a))$, $a \neq 0$, is on the graph of odd function $y = g(x)$. What 2 points are guaranteed to be in the graph?

5. Explain how it is possible for an infinite number of functions to be **both** odd and even.

6. Give an example to show that the sum of 2 non-odd functions can be odd.

7. Classify each of the six trigonometry functions as being either "odd" or "even".

8. Explain why $y = \log x^2$ is even but $y = 2 \log x$ is not.

9. Classify each of these principal-valued inverse trigonometric functions as "odd", "even", or "neither", or "both"

(a) $y = \sin^{-1}x$ (b) $y = \cos^{-1} x$ (c) $y = \tan^{-1}x$

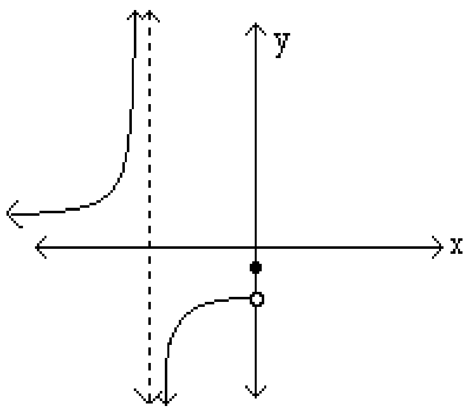
10. Write definitions so that one can determine if $f \circ g$ is an even or odd function.

11. Let f be an even function and g be an odd function. Classify as "odd" or "even".

(a) $f \circ f$ (c) $f \circ g$

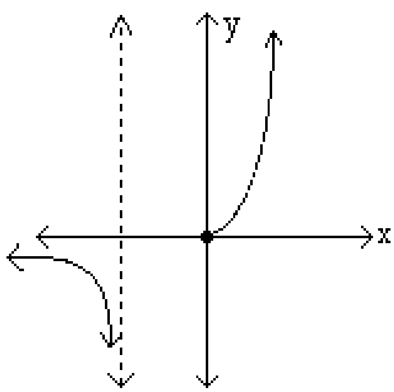
(b) $g \circ g$ (d) $g \circ f$

12.



Given: f is an even function.
Complete the graph.

13.



Given: f is an odd function.
Complete the graph.

14. What restrictions on the domain of f would enable $f(x) = x^5 - 4x^3$ to be an even function? ✍