PERFECT SQUARES, FIBONACCI SEQUENCES, PATTERNING & CONNECTING -- Explorations For The Classroom

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In Problem Solving we were looking at the following pattern:

$$0 \cdot 2 + 1 = 1$$
$$1 \cdot 3 + 1 = 4$$
$$2 \cdot 4 + 1 = 9$$
$$3 \cdot 5 + 1 = 16$$
$$4 \cdot 6 + 1 = 25$$

The class quickly generalized for the nth line and came up with:

$$(n - 1)(n + 1) + 1 = n^2 - 1 + 1 = n^2.$$

Several weeks later we were considering the pattern:

$$0 \cdot 1 \cdot 2 \cdot 3 + 1 = 1 = 1^2$$
$$1 \cdot 2 \cdot 3 \cdot 4 + 1 = 25 = 5^2$$
$$2 \cdot 3 \cdot 4 \cdot 5 + 1 = 121 = 11^2$$
$$3 \cdot 4 \cdot 5 \cdot 6 + 1 = 361 = 19^2$$
$$4 \cdot 5 \cdot 6 \cdot 7 + 1 = 841 = 29^2$$
$$5 \cdot 6 \cdot 7 \cdot 8 + 1 = 1681 = 41^2$$

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At this point students were asked to conjecture what the next five terms would be in the sequence:

$$1^2, 5^2, 11^2, 19^2, 29^2, 41^2$$

Most students looked at the sequence

1 5 11 19 29 41
  4 6 8 10 12

with consecutive differences 4, 6, 8, 10, 12 to produce the next five terms: $55^2, 71^2, 89^2, 109^2, 131^2$.

One student, Mark, looked at the data and said "It's obvious that the number produced in any line is a square and the formula is easy to obtain."

Here was Mark's argument:

The nth row can be expressed by writing: $(n - 1)(n)(n + 1)(n + 2) + 1$

By pairing the 1st and 4th factors and 2nd and 3rd factors we have:

$$(n - 1)(n + 2)n(n + 1) + 1$$
$$= (n^2 + n - 2)(n^2 + n) + 1$$

which is the same form as that of the first pattern determined several weeks earlier.

Thus, $(n^2 + n - 2)(n^2 + n) + 1 = [(n^2 + n - 1) - 1][(n^2 + n - 1) + 1] + 1$ is always a square for n = 0, 1, 2, 3, . . . and it is equal to $[(n^2 + n - 2) + 1]^2$ or $(n^2 + n - 1)^2$.

Thus, $(n - 1)(n)(n + 1)(n + 2) + 1 = (n^2 + n - 1)^2$.

At a later point in the course we were generating Fibonacci type sequences beginning with two different seed numbers.

	Seed	Nos.								
Sequence #1	1	2	3	5	8	13	21	34	55	. . .
Sequence #2	1	3	4	7	11	18	29	47	76	. . .
Sequence #3	1	4	5	9	14	23	37	60	97	. . .
Sequence #4	1	5	6	11	17	28	45	73	118	. . .
Sequence #5	1	6	7	13	20	33	53	86	139	. . .

[Note: What patterns can you find?]

We were investigating the following:

1. Take any term, S_n, $n \geq 2$, of the above sequences.
2. Evaluate: $\left| (S_n)^2 - (S_{n-1})(S_{n+1}) \right|$.

   What do you observe?

My students quickly discovered that choosing any element in:

Sequence #1 yields a value of 1.
Sequence #2 yields a value of 5.
Sequence #3 yields a value of 11.
Sequence #4 yields a value of 19.

We observed that the sequence 1, 5, 11, 19, . . . seems to be connected with an earlier sequence. It is left as a matter of exploration to reconcile this observation. ✍