

**PERFECT SQUARES, FIBONACCI SEQUENCES,
PATTERNING & CONNECTING -- Explorations For The Classroom**

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In Problem Solving we were looking at the following pattern:

$$\begin{aligned}0 \cdot 2 + 1 &= 1 \\1 \cdot 3 + 1 &= 4 \\2 \cdot 4 + 1 &= 9 \\3 \cdot 5 + 1 &= 16 \\4 \cdot 6 + 1 &= 25\end{aligned}$$

The class quickly generalized for the n th line and came up with:

$$(n - 1)(n + 1) + 1 = n^2 - 1 + 1 = n^2.$$

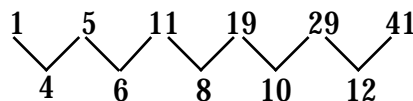
Several weeks later we were considering the pattern:

$$\begin{aligned}0 \cdot 1 \cdot 2 \cdot 3 + 1 &= 1 = 1^2 \\1 \cdot 2 \cdot 3 \cdot 4 + 1 &= 25 = 5^2 \\2 \cdot 3 \cdot 4 \cdot 5 + 1 &= 121 = 11^2 \\3 \cdot 4 \cdot 5 \cdot 6 + 1 &= 361 = 19^2 \\4 \cdot 5 \cdot 6 \cdot 7 + 1 &= 841 = 29^2 \\5 \cdot 6 \cdot 7 \cdot 8 + 1 &= 1681 = 41^2 \\&\cdot \\&\cdot \\&\cdot\end{aligned}$$

At this point students were asked to conjecture what the next five terms would be in the sequence:

$$1^2, 5^2, 11^2, 19^2, 29^2, 41^2$$

Most students looked at the sequence



with consecutive differences 4, 6, 8, 10, 12 to produce the next five terms:

$$55^2, 71^2, 89^2, 109^2, 131^2.$$

One student, Mark, looked at the data and said "It's obvious that the number produced in any line is a square and the formula is easy to obtain."

Here was Mark's argument:

The n th row can be expressed by writing: $(n - 1)(n)(n + 1)(n + 2) + 1$

By pairing the 1st and 4th factors and 2nd and 3rd factors we have:

$$\begin{aligned}&(n - 1)(n + 2)n(n + 1) + 1 \\&= (n^2 + n - 2)(n^2 + n) + 1\end{aligned}$$

which is the same form as that of the first pattern determined several weeks earlier.

Thus, $(n^2 + n - 2)(n^2 + n) + 1 = [(n^2 + n - 1) - 1][(n^2 + n - 1) + 1] + 1$ is always a square for $n = 0, 1, 2, 3, \dots$ and it is equal to $[(n^2 + n - 2) + 1]^2$ or $(n^2 + n - 1)^2$.

Thus, $(n - 1)(n)(n + 1)(n + 2) + 1 = (n^2 + n - 1)^2$.

At a later point in the course we were generating Fibonacci type sequences beginning with two different seed numbers.

	<u>Seed</u>	<u>Nos.</u>								
Sequence #1	1	2	3	5	8	13	21	34	55	...
Sequence #2	1	3	4	7	11	18	29	47	76	...
Sequence #3	1	4	5	9	14	23	37	60	97	...
Sequence #4	1	5	6	11	17	28	45	73	118	...
Sequence #5	1	6	7	13	20	33	53	86	139	...

[Note: What patterns can you find?]

We were investigating the following:

1. Take any term, S_n , $n \geq 2$, of the above sequences.
2. Evaluate: $(S_n)^2 - (S_{n-1})(S_{n+1})$.

What do you observe?

My students quickly discovered that choosing any element in:

Sequence #1 yields a value of 1.
 Sequence #2 yields a value of 5.
 Sequence #3 yields a value of 11.
 Sequence #4 yields a value of 19.

We observed that the sequence 1, 5, 11, 19, ... seems to be connected with an earlier sequence. It is left as a matter of exploration to reconcile this observation. 🐣