INTEGRATING WITH LIMITS: THE PHYSICS VIEW

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One of the ways that physicists and mathematicians differ in their respective use of calculus is that math people usually integrate without limits while physics people almost always integrate with limits. In physics, the limits play an often crucial role in understanding the results of the integration. Here's a simple example from kinematics. Suppose an object is moving with a uniform acceleration, **a**. Since acceleration is the time rate-of-change of velocity, $\frac{dv}{dt}$, we can write:

$$\frac{dv}{dt} = a \qquad \text{or} \qquad dv = a\,dt$$

integrating, we have

$$\int dv = \int a\,dt$$

If we perform this integration without limits, we will get an arbitrary constant

$$v = at + c$$

and it is difficult to assign a physical significance to this constant. But, suppose we decide to integrate over the definite time interval from 0 to t seconds. The object's velocity was v_0 at $t = 0$ and became v at t = t. Therefore,

$$\int_{v_0}^{v} dv = \int_0^t a\,dt \qquad \text{which after integration becomes}$$

$$v - v_0 = a\,(t - 0) \qquad \text{or}$$

$$v = v_0 + at \tag{1}$$

Now, the meaning of the constant becomes clear; it's just the initial velocity, v_0. Similarly, if we look at the object's displacement, x, we see that since $v = \frac{dx}{dt}$,

$$\int dx = \int v\,dt$$

If we integrate over the same definite time interval as we did before (0 to t), we have the object at x_0 when $t = 0$, and at x when t = t. So,

$$\int_{x_0}^{x} dx = \int_0^t v\,dt \qquad \text{substituting for v from equation (1)}$$

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + at)\,dt$$

integrating, we have

$$x = x_0 + v_0\,t + \frac{1}{2}\,at^2. \tag{2}$$

Again, the constant for equation (2) comes directly from the integration limits.

But the use of definite integrals can do much more than supply meaningful integration constants. It can sometimes provide an unexpected insight into the workings of Nature.

Consider the following situation. A point mass, **m**, is placed a distance, **r**, from the center of a hollow thin shell of mass, **M**., as shown in Figure 1 below.

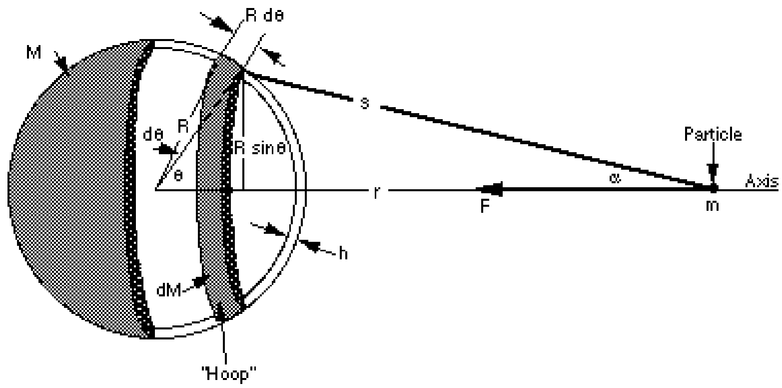

Figure 1

We want to find the gravitational force of attraction between the shell and **m**. The shell has a radius, **R**, and is **h** thick. One way to solve this problem is to imagine the shell divided into a series of "hoops." Each hoop is **h** thick and **R d**θ wide.

If we consider any small piece of the hoop, **dM**, say a piece at the top, it would exert a force on **m** along **s** equal to $\frac{G\,dM\,m}{s^2}$.[1] But, since there always exists another piece of hoop diametrically opposite (in this case, at the bottom of the hoop) the components of force transverse to the axis cancel out, and we need consider only the component along the axis: $F_{net} = \frac{G\,dM\,m}{s^2}\cos\alpha$. All the differential pieces of an individual hoop contribute equally so we can let **dM** represent the mass of the entire hoop. Since the shell is of uniform thickness, the mass of any hoop, **dM**, is proportional to its volume, **dV**.

$$\frac{dM}{M} = \frac{dV}{V}$$

The volume of the hoop in the figure is its thickness, **h**, times its width, **R d**θ, times its circumference, $2\pi R \sin\theta$. The volume of the entire shell, **V**, is $4\pi R^2 h$. So

$$dM = \frac{M\,dV}{V} = \frac{M \cdot 2\pi R^2 h \sin\theta\, d\theta}{4\pi R^2 h} = \frac{M}{2}\sin\theta\, d\theta$$

A hoop, therefore, pulls on the mass, m, with a force

$$dF = \frac{G\,M\,m}{2s^2}\cos\alpha \sin\theta\, d\theta$$

Now, all we have to do is add up all the hoops that comprise the shell.

[1] This is Newton's gravitational force law. The force between two point masses varies directly with the size of the masses and varies inversely with the square of the separation distance.

$$\mathbf{F} = \int \mathbf{dF} = \frac{\mathbf{G\,M\,m}}{\mathbf{2}} \int \frac{\mathbf{cos}\,\alpha\ \mathbf{sin}\,\theta\,\mathbf{d}\theta}{\mathbf{s^2}} \tag{3}$$

It's a little difficult to assign limits to this integral because it looks as if we are integrating three different variables: the angles α and θ and the distance, **s**. But these three variables are *not* independent of one another. Look at the triangle formed by **r**, the distance from the center of the shell to the particle, **s**, the distance from the hoop to the particle, and **R**, the radius of the shell.

If we use the law of cosines we have,

$$\mathbf{s^2 = R^2 + r^2 - 2Rr\cos}\,\theta$$

Since r and R are constant, we can easily take the differential of this equation to get,

$$\mathbf{2s\,ds = 2Rr\sin}\,\theta\,\mathbf{d}\theta$$

or

$$\mathbf{sin}\,\theta\,\mathbf{d}\theta = \frac{\mathbf{s\,ds}}{\mathbf{Rr}} \tag{4}$$

That worked so well, let's try using the law of cosines again on the same triangle.

$$\mathbf{R^2 = s^2 + r^2 - 2sr\cos}\,\alpha$$

or

$$\mathbf{cos}\,\alpha = \frac{\mathbf{s^2 + r^2 - R^2}}{\mathbf{2sr}} \tag{5}$$

Let's substitute equations 4 and 5 into equation 3.

$$\mathbf{F} = \frac{\mathbf{GMm}}{\mathbf{2}} \int \frac{\mathbf{s^2 + r^2 - R^2}}{\mathbf{2Rr^2s^2}}\,\mathbf{ds}$$

We can separate this expression into two integrals.

$$\mathbf{F} = \frac{\mathbf{GMm}}{\mathbf{4Rr^2}} \int \mathbf{ds} + \frac{\mathbf{GMm(r^2\text{-}R^2)}}{\mathbf{4Rr^2}} \int \mathbf{s^{-2}\,ds}$$

And now, with s as the only variable, we can determine limits. We want to add up the hoops from the near end of the shell, r – R, to the far end of the shell, r + R. So,

$$\mathbf{F} = \frac{\mathbf{GMm}}{\mathbf{4Rr^2}} \int_{\mathbf{r\text{-}R}}^{\mathbf{r+R}} \mathbf{ds} + \frac{\mathbf{GMm(r^2\text{-}R^2)}}{\mathbf{4Rr^2}} \int_{\mathbf{r\text{-}R}}^{\mathbf{r+R}} \mathbf{s^{-2}\,ds} \tag{6}$$

After integration we have

$$\mathbf{F} = \frac{\mathbf{GMm}}{\mathbf{4Rr^2}}\,\mathbf{(2R)} + \frac{\mathbf{GMm(r^2\text{-}R^2)}}{\mathbf{4Rr^2}} \left(\frac{\mathbf{1}}{\mathbf{r\text{-}R}} - \frac{\mathbf{1}}{\mathbf{r+R}} \right)$$

or

$$\mathbf{F} = \frac{\mathbf{GMm}}{\mathbf{2r^2}} + \frac{\mathbf{GMm}}{\mathbf{2r^2}} = \frac{\mathbf{GMm}}{\mathbf{r^2}}$$

This is Newton's law of gravitational force between two point masses **m** and **M**, a distance, **r**, apart. In other words, it's as if all the mass in the shell were concentrated at the geometric center of the shell. This may be interesting, but it's hardly surprising, and certainly not "an unexpected insight into the workings of Nature." The insight comes when you ask the question, "What happens if we move the particle to a point inside the shell -- not at the center which is a special case, but to *any other* point inside?" Figure 2 shows this situation.

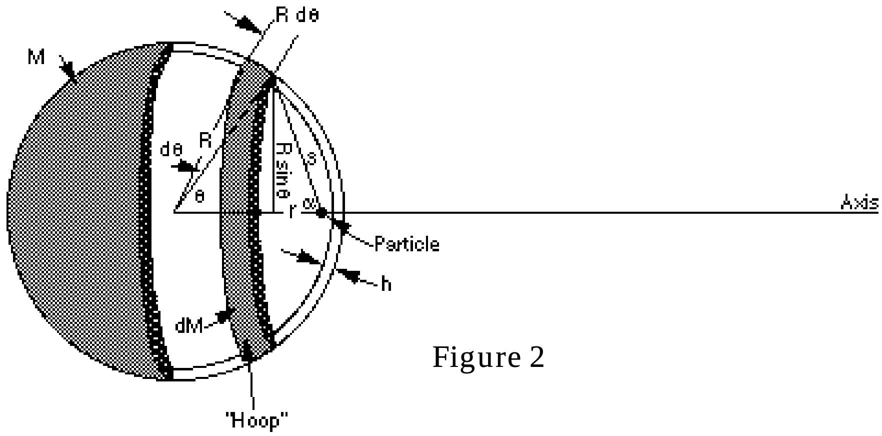

Figure 2

If you look at the diagram, you'll see that nothing has really changed in our analysis. We can still divide the shell into hoops. Each hoop still has the same effect on the particle. The only difference lies in the *limits*. If the particle is placed inside the shell, then when we add up the hoops from one end of the shell to the other, we're summing from **R – r** to **R + r**. Equation 6, our integral for total force now looks like

$$F = \frac{GMm}{4Rr^2}\int_{R-r}^{R+r} ds + \frac{GMm(r^2-R^2)}{4Rr^2}\int_{R-r}^{R+r} s^{-2}\, ds$$

After integration we have

$$F = \frac{GMm}{4Rr^2}(2r) + \frac{GMm(r^2-R^2)}{4Rr^2}\left(\frac{1}{R-r} - \frac{1}{R+r}\right)$$

So,

$$F = \frac{GMm}{2rR} - \frac{GMm}{2rR} = 0$$

In other words, there is no gravitational force on a mass *anywhere* inside a hollow shell. This is an insight. And it applies not only to gravitational forces but also to *any* force whose intensity varies inversely as the square of distance. For example, the electric force. Our insight here predicts that there would be no electric force on a charge inside a hollow, charged, conducting sphere. And all from integrating with limits. ✍