THE EULER FORMULA—PART I

by: Dr. Micah Fogel
Illinois Mathematics and Science Academy

The great Swiss mathematician Leonhard Euler (1707–1783, pronounced like "oiler") discovered a geometric formula that now bears his name. It relates the number of sides and the number of corners of various geometric figures. The formula has a variety of applications. In this article we will develop the formula for figures that are made up entirely out of straight lines. Later in this article we will adapt the formula to figures whose sides can be curved, and apply it to solve a problem that is sometimes called "water, gas, and electricity."

Polygons

Consider any polygon, P, in the plane (see Figure 1). It is clear that P has the same number of sides as it has corners. In order to make our language consistent when we talk about three-dimensional objects, we will use the word *vertex* to mean "corner", and the word *edge* for "side." Thus an edge is a line segment in the figure and a vertex is where edges meet. If V is the number of *vertices* (plural of vertex) in our polygon, and E is the number of edges, then $E = V$, or $E - V = 0$.

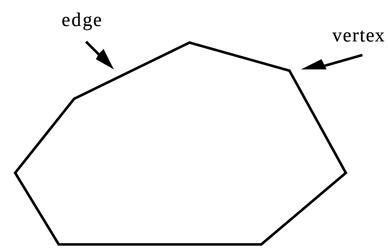


Figure 1. A polygon P.

We can actually prove that this formula will always be true. It is obviously true for a triangle. Since any polygon can be split up into small pieces (a process called "subdivision") each of which is a triangle (see Figure 2), any polygon can be thought of as consistency of a number of triangles glued together along their edges.

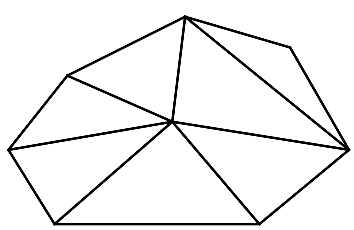


Figure 2. Subdividing the polygon.

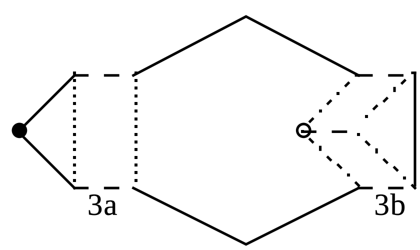


Figure 3. Gluing in a triangle. The dotted edges get obliterated in the process. The circled vertex gets obliterated, while the heavy dot indicates a new vertex.

So let's say you are making up a polygon by gluing triangles together, and you are ready to glue in one more triangle. There are two possibilities, corresponding to Figures 3a and 3b. In 3a, one edge of the new triangle is glued onto the old collection of triangles. The gluing will add one new vertex to the figure and two new edges. But it also obliterates one of the edges of the old collection, the one along which the triangle was glued. So the net effect is to add one more vertex and one more edge. So V and E each increase by one, and $E - V$ still equals zero.

In 3b, the new triangle is glued into the old collection along *two* edges. This time, no new vertices are added, and in fact one of the old vertices is removed, because it will now be entirely inside the collection. One new edge is added, but two old edges are taken away, since we are gluing along two edges. So the net effect is to decrease both V and E by one, still leaving $E - V$ unchanged.

There can't be a case where you glue in the new triangle along all three of its sides. Because then there had to be a hole in the middle of the original collection, and polygons don't have holes in them!

Explorations

What happens in case you do allow your polygons to have holes in them? Draw some polygons that have holes in them and count their edges and vertices. A polygon with a hole *cannot* be created by gluing triangles together one at a time, since gluing in the triangle that creates the hole will result in a shape like the one on the left in Figure 4, which is not a polygon. A hole can be created by gluing in a *square* as in Figure 5. This does not change the number of vertices or edges in the figure.

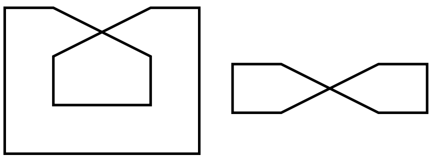


Figure 4. Not polygons!

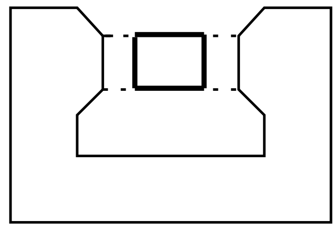


Figure 5. Gluing in a rectangle to create a hole.

What goes wrong with Euler's formula for the shapes in Figure 4? Notice that in a polygon (even one with a hole in it) there are always exactly two edges that meet at each vertex.

Polyhedra

Now we will examine what happens in three dimensions. The analog of a polygon in three dimensions is an object called a *polyhedron*. A polyhedron is simply a three-dimensional shape with all flat sides. Each side will be called a *face*. The edges and vertices of a polyhedron are the line segments and points where two or more faces meet. For example, a cube has six faces, twelve edges, and eight vertices.

In analogy with the fact that polygons don't have holes in them, we require that our polyhedra be solid, that is, they have no hollow spaces inside them. We also prohibit them from having holes drilled through them, so square donuts are out.

The simplest kind of polyhedron is called a tetrahedron, because it has four faces (see Figure 6). It clearly has six edges and four vertices. What happens when two of these are glued together face-to-face (Figure 7)? You get an object with six faces, five vertices, and nine edges. In fact, the number of edges increased by the same number that faces and vertices increased, combined.

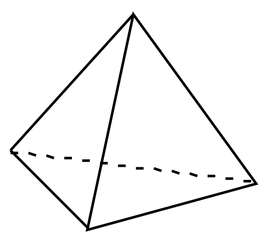


Figure 6. A tetrahedron.

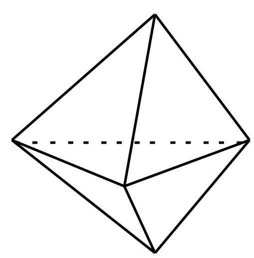


Figure 7. Two tetrahedra glued together.

This will always be true. If you have a bunch of tetrahedra glued together, and you want to glue on one more, there are three ways to do it: along one face (Figure 8a), along two faces (Figure 8b) and along three faces (Figure 8c). If it were glued in along all four faces, then it would be plugging a hollowed out space, which we don't allow in our polyhedra. In the first case, the number of edges increases by three, faces by two, and vertices by one. In the second case, none of the three numbers changes. In the third case, the number of edges decreases by three, faces by two, and vertices by one. So the change in the number of edges always equals the combined change in vertices plus faces. So we might examine $V + F - E$.

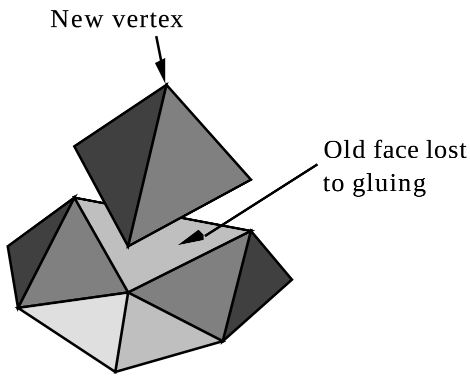


Figure 8a. Gluing a tetrahedron along one face.
One new vertex, three new edges,
three new faces, one old face lost to gluing.

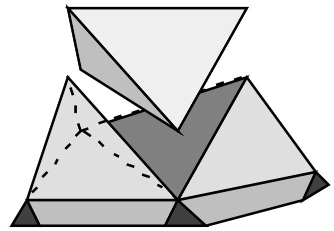


Figure 8b. Gluing along two faces.
One new edge and two new faces
new edge and two new faces, one
old edge and two faces lost to gluing.

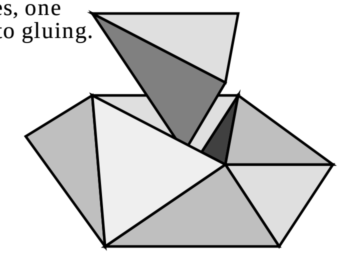


Figure 8c. Filling in a depression by
gluing along three faces. One vertex,
three edges, and three faces disappear,
as one new face is gained.

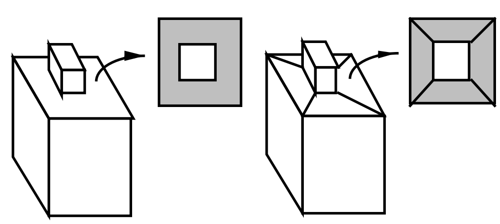
What Euler discovered is that for any polyhedron that has no hollowed out spaces and no holes, $V + F - E = 2$.

This is true for the tetrahedron ($4 + 4 - 6 = 2$) and the cube ($8 + 6 - 12 = 2$). We can prove it for any polyhedron by splitting it into a collection of tetrahedra glued together. The formula works for the first tetrahedron, and then each other tetrahedron is glued on as in Figure 8a, b, or c, and we already know that $V + F - E$ won't change if we glue in these ways.

Exploration: Familiar Shapes

Examine Euler's formula as it relates to some familiar shapes, like boxes and pyramids How many seams are there on a soccer ball? It has 12 pentagonal faces and 20 hexagonal faces, and each vertex belongs to exactly one of the pentagons, so there are 60 vertices.

When examining complicated objects, keep in mind that every face of a poly-hedron must itself be a polygon, so it can't have any holes in it. See what happens for a cube glued onto the side of a larger cube in Figure 9.



Wrong: 11 faces, 24 edges, 16 vertices; top of large cube is not a polygon.

Right: Top of big cube broken down into four polygons. 14 faces, 28 edges, 16 vertices.

Figure 9. Each face must be a polygon with no holes!

Exploration: Curved Shapes

Like the soccer ball in the exploration above, most of the objects we deal with are not made up entirely of flat edges. But if they are “close” then Euler’s formula still applies, as it did above. “Close” means that if the object were made out of very stretchy rubber, it could be bent into a polyhedron. Still keeping in mind that the faces of polyhedra must be polygons, explore other curved shapes like basketballs or cylinders. Note that you must have edges running up and down the side of the cylinder or you won’t be able to bend the side into a polygon.

Exploration: The Platonic Solids

The cube has all faces made of regular polygons, all of the dihedral angles where two faces meet are the same, and around every vertex there are the same number of faces. There are only four other shapes that can have the same said about them—the regular tetrahedron (four triangular faces), octahedron (eight triangular faces), dodecahedron (twelve pentagonal faces) and icosahedron (twenty triangular faces). These are called the regular or Platonic solids (see Figure 10). Other shapes (like a wargamer’s ten-, thirty-, or hundred-sided dice) fail in some way to be as perfect as the Platonic solids.

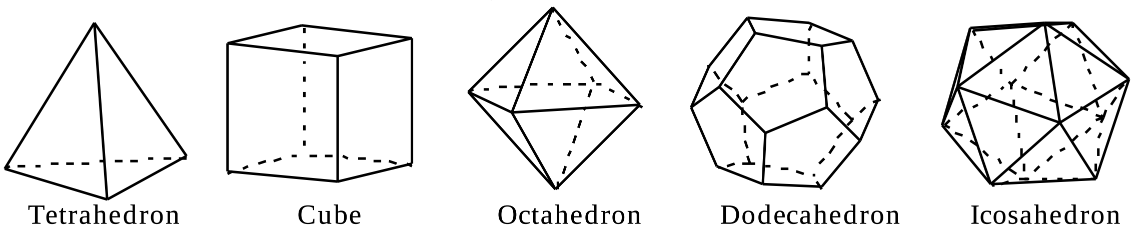


Figure 10. The Platonic solids.

Use Euler’s formula to prove that there can’t be any other perfect solids. Hints: each face must have the same number of edges, each edge is shared by exactly two faces, and every vertex has the same number of edges emanating from it. The solution is given at the end of the article.

Exploration: Holes

What if we want our polyhedra to be able to have holes in them, like Figure 11? The way we do this is to let two pieces of a shape without holes *grow* toward each other until they finally meet, and then glue the pieces together. If they are glued together along a polygon with n edges, then the gluing oblitgerates n edges, n vertices, and two faces . So $V + F - E$ decreases by two.

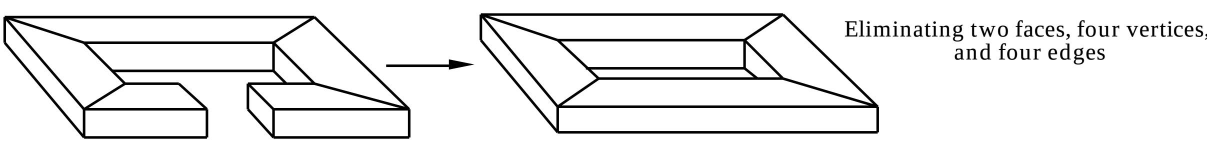


Figure 11. Creating a hole.

Convince yourself that if you make g holes in this manner, the resulting shape has $V + F - E = 2 - 2g$. The number g, which is the number of holes in your shape, is called the *genus* of the shape.

Solution: Platonic Solids

Each face must be a regular polygon with n sides, and around each vertex, exactly m of these faces must be assembled. Since each edge is shared by exactly two faces, there are exactly $E = nF / 2$ edges. Each edge has two ends, which are vertices, and exactly m edges come together at each vertex, so there are $V = nF / m$ vertices. You also have Euler's formula $V + F - E = 2$. Thus, you can solve for F in terms of m and n. So once you know m and n you know all about the regular polygon that they determine.
m and n are restricted by the facts that the polygons have n sides, and that $360 / m$ is larger than the number of degrees in each interior angle of a regular polygon of n sides (as we are trying to assemble m of them around a point).

Think about the restrictions that are placed on m and n. The only combinations possible are:

- $n = 3$, $m = 3$: this yields the tetrahedron with $F = 4$, $E = 6$, and $V = 4$
- $n = 3$, $m = 4$: this yields the octahedron with $F = 8$, $E = 12$, and $V = 6$
- $n = 3$, $m = 5$: this yields the icosahedron with $F = 20$, $E = 30$, and $V = 12$
- $n = 4$, $m = 3$: this yields the cube with $F = 6$, $E = 12$, and $V = 8$
- $n = 5$, $m = 3$: this yields the dodecahedron with $F = 12$, $E = 30$, and $V = 20$

The Euler Formula—Part II

In Part I of this article, we developed the Euler formula which says that $E - V = 0$ for any polygon and $F - E + V = 2$ for any polyhedron. The only restrictions on the figures were that there were no holes in them, that every face of a polyhedron had to be a polygon, and that all the faces were flat and edges straight. An exploration there in Part I suggested that it would be possible to extend the formula to cases where things were curved. This is indeed possible, and this article will explain the details. This article will end by applying the results to solve the classic problem known as "water, gas, and electricity."

Curved Polygons

Many familiar shapes from geometry aren't made of all straight lines and flat faces, like circles, cones, spheres, and the like. We would like Euler's formula to apply to them as well. And it does, if we just allow our edges and faces to be curved. Just like in the case of flat and straight objects, we will start with polygons.

What should a curved polygon look like? Basically, we will call a figure a polygon if the sides can be "straightened out" to yield a straight polygon. See Figure 1. (Aside from calculus students: the exact definition is that a curved polygon is any figure which is the image of a circle under a piecewise differentiable one-to-one function.)

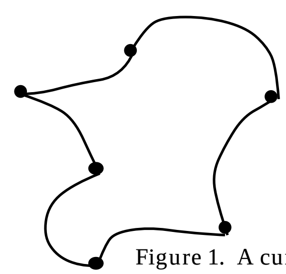


Figure 1. A curved polygon.

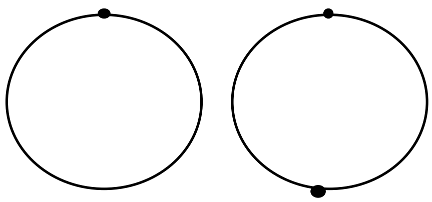


Figure 2. A monogon and a digon.

There are two special curved polygons that have no analogs among polygons with straight edges. They are the one-sided monogon and the two-sided digon (see Figure 2). They are unusual because with straight sides, you cannot make up something that circles back on itself with fewer than three sides.

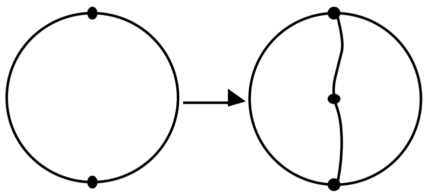


Figure 3. Subdividing a digon into two "triangles."

The way we proved that the Euler formula worked for straight edged polygons was to subdivide the polygon into triangles. The same proof works here, except in the case of the monogon (a digon can be subdivided into two curved triangles—see Figure 3). But a monogon has one edge and one vertex, so the formula works for monogons as well.

Exploration: Curved Polygons

Draw any closed curve in the plane that does not cross itself (a “simple” closed curve). Put any number of points anywhere on the curve, and call these points the vertices. The vertices cut the curve into pieces, so call these pieces edges. Verify Euler’s formula for this curve.

Curved Polyhedra

What should a curved polyhedron look like? It should be allowed to look like any object in everyday life that does not have any holes in it. So spheres, cones, tables, and salad forks should all be examples of curved polyhedrons. Any object that could be made with a lump of super stretchable clay without punching a hole in it or tearing it into two or more lumps is a curved polyhedron. (Aside from advanced calculus students: the exact definition is that a curved polyhedron is the image of a sphere under a piecewise differentiable one-to-one function.)

Normal polyhedra always have faces, edges, and vertices, and in fact the entire outside surface of the polyhedron is split up into polygons which formed the faces of the polyhedron. The same must be true of a curved polyhedron. The surface of a curved polyhedron is partitioned into shapes that are curved polygons, and these polygons are called the faces of the curved polyhedron. The curved sides of these faces where two faces are joined are the edges of the polyhedron, and the points where edges come together are the vertices.

For instance, a soccer ball is a curved polygon. It's faces are the 32 curved hexagons and pentagons that are the black and white patches that make up the ball. Its edges are the 90 seams that hold the ball together, and its vertices are the points where seams come together. There are 60 of them.

Figure 4 shows a cone. It has seven faces (the six faces that make up the cone plus the six-sided base), seven vertices (six on the base and one at the top) and twelve edges (six running along the cone and six around the base).

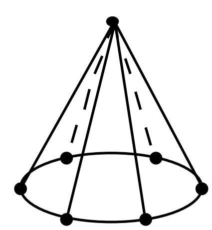


Figure 4. A curved cone with six "sides".

Exploration: Curved Polyhedra

Take some simple, common shapes, such as spheres and cylinders. Divide up their surface areas so that each division is a curved polyhedron. Count the number of faces, edges, and vertices you obtained. Did these numbers satisfy the Euler formula $F + V - E = 2$? If not, are you sure that your object has no holes? Are you sure the polygons you used for your division don’t have any holes?

Exploration: The Problem With Holes:

Divide a sphere as shown in Figure 5. This is like dividing a globe into the tropics and the polar areas north and south of the tropics. If you count faces, edges, and vertices in this figure, you will notice that there are eight vertices (four in each polar region), twelve edges (six in each polar region), and *seven* faces (three in each polar region, plus the one that makes up the tropics). So $F + V - E = 1$, apparently defying the Euler formula. What went wrong? The "face" that makes up the tropics really isn't a curved polygon, so this isn't a "legal" way of splitting up the surface of a globe.

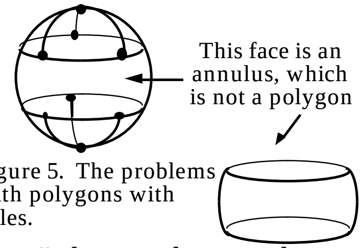


Figure 5. The problems with polygons with holes.

The Euler Formula

The Euler formula works for curved polyhedra just as it does for flat ones. For exactly the same reason, any curved polyhedron can be obtained from gluing together curved tetrahedra. The proof works the same as for flat polyhedra as long as all the faces have at least three edges (in other words, digon or monogon faces cause problems).

Digons do not really cause problems, because if you pick any point on the perimeter of the digon that is not already a vertex, and just declare that it is a vertex, you have turned a digon into a three-sided polygon. See Figure 6, where a sphere had originally been paritioned into two digons, but after declaring a new point to be a vertex, it became a partition into two three-sided figures . The reason this trick works is that this creation of a new vertex adds one vertex, splits one edge into two, and creates no new faces, so that $V + F - E$ remains unchanged.

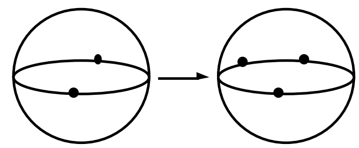


Figure 6. Digons can be turned into three-sided figures.

The problem with monogons can be fixed with a slightly different trick, which is explained in Figure 7. A new face is created which wraps around the monogon. Two new vertices and three new edges are created, along with one new face. Again, $V + F - E$ remains unchanged. After this new face is created, the monogon can be deleted, which removes one edge, and merges two faces into one, so $V + F - E$ is still unchanged. With these tricks, all the monogons and digons can be replaced by a figure with at least three sides, and then the old proof of the Euler formula works.

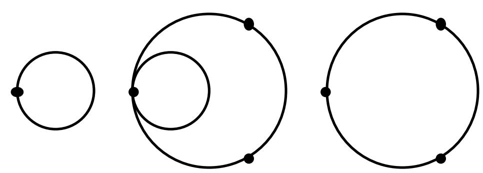


Figure 7. Replacing a monogon with a three-sided polygon.

Water, Gas, and Electric

A classic puzzle is to draw three houses on a piece of paper, and then draw three utilities, water, gas, and electric. Can you connect each house to each utility without crossing any lines (see Figure 8)?

The next few paragraphs will provide the solution, so you may wish to try it on your own before reading on.

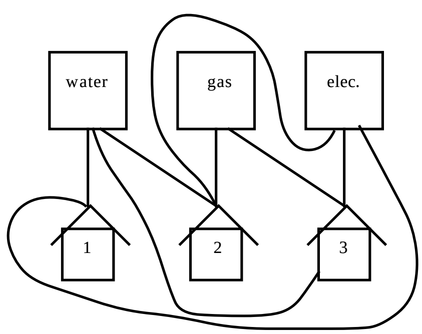


Figure 8. Not quite a solution; house 1 has no gas. Can every house be connected to all utilities?

The Solution

We will actually demonstrate that this problem can't be solved.

Assume you have drawn a solution. Draw your solution on the surface of a sphere (this is why we need curved shapes). Your solution forms a bunch of edges and vertices on the sphere. There are six vertices (three houses and three utilities) and nine edges (each house is connected to each utility). What kind of faces are the result of this division?

First, if a vertex represents a house, the vertices it is connected to are all utilities, and vice versa. Thus any face must have an even number of edges. And since you never need the same utility and house to be connected to each other twice, you don't have any faces with only two edges. So all faces have four or six vertices.

In addition, no face has a hole in it, for along the outside of any face there are at least four vertices, and if there is a hole inside the face, this hole is also a face with at least four vertices. But there are only six vertices!

Since none of the faces have holes, Euler's formula is true, and with nine edges and six vertices, you must have five faces. Each face has at least four edges, so if you total the number of edges in all faces, you get at least twenty. This counts each edge twice (each edge is a part of two connected faces), so there must be at least ten edges to account for having five faces. But there are only nine edges! This is a contradiction, so the assumption that you have drawn a solution must be incorrect. There is no solution to the water-gas-electric problem.

Explorations

Can you solve the water-gas-electric problem on the surface of a donut?

What does Euler's formula say about Möbius strips?

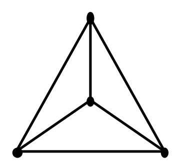


Figure 9. Four points can be connected together without crossing lines. Can five?

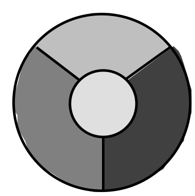


Figure 10. At least four colors are needed to color a picture without contiguous areas being the same color.

Figure 9 shows that you can draw four points on the plane, each of which is connected to each of the others. Can you do this with five points without crossing any lines? If you draw a map on the plane, the *Four Color Map Theorem* says you never need more than four colors to color your map so that no two regions with a common border have the same color. Figure 10 is closely related to Figure 9 and shows that at least four colors are really necessary. If you were able to draw five connected points for the previous question, you will be able to draw five countries that each touch each other, and hence need five colors (so you will have disproved the *Four Color Map Theorem*). More likely, you proved that you couldn't draw five such points. Does this prove the Four Color Map Theorem? Why or why not? ✍