

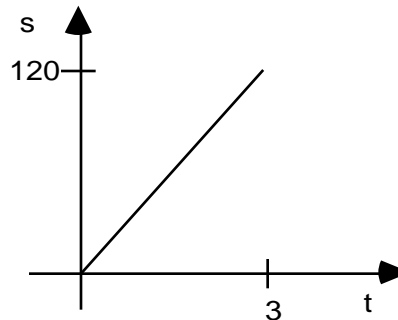
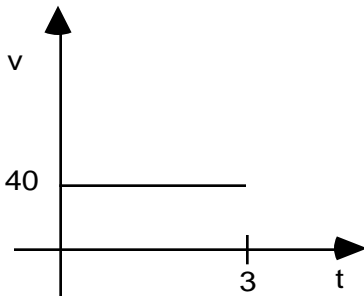
EULER'S METHOD

by: Ruth Dover

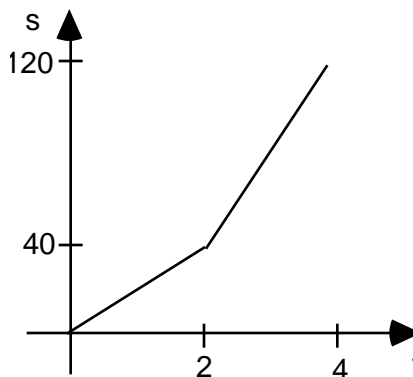
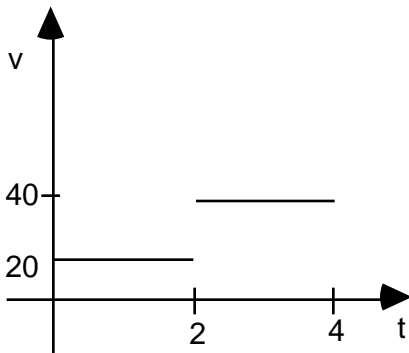
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Leonhard Euler, a Swiss mathematician, offered a method for creating graphical solutions to differential equations. It appears near the end of some calculus books. Others have skipped the topic entirely. I recently re-discovered this topic and introduced it in my Calculus I course before even mentioning that magic word "derivative!"

We began the semester by talking about position and velocity. More generally, we talked about a function and its rate of change. We drew several possible graphs of functions from written descriptions of situations. Then we drew associated rate graphs. Very often, the topics dealt with motion in a car and the associated speed or velocity. We examined the specific question of finding the position function, assuming that one was given the velocity or rate of change function. If the velocity is continuously changing, this requires a good understanding of the notions of the derivative and anti-derivative. However, if the velocity is constant, then the process of finding the position function $s = s(t)$ becomes very straightforward. For example, if the velocity is given as 40 mph, and the elapsed time is 3 hours, then the total distance traveled will be $40 \text{ mph} \cdot 3 \text{ h} = 120 \text{ m}$. If we assume that we start at $s(0) = 0$, the graphs are as follows:

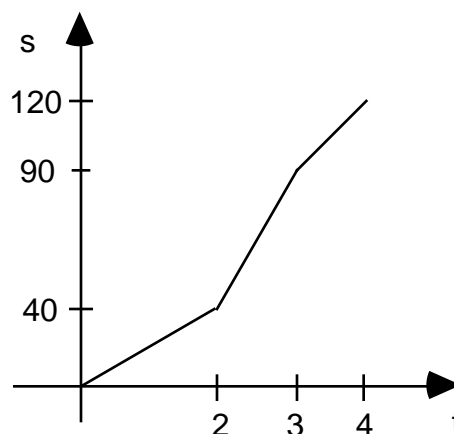
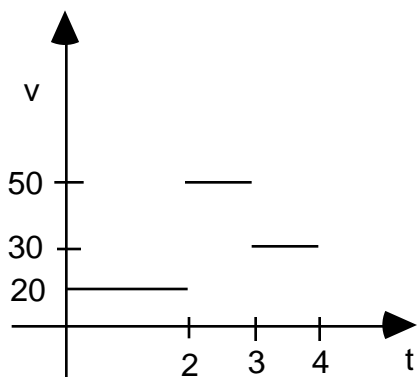


But what if two different velocities are involved? The process is essentially the same, though we must be sure to add the two distances together to find the total distance traveled.



In the first two hours, the car travels $20 \text{ mph} \cdot 2 \text{ h} = 40 \text{ m}$. In the second two hours, the car travels $40 \text{ mph} \cdot 2 \text{ h} = 80 \text{ m}$. The combined distance traveled is $40 + 80 = 120 \text{ m}$.

We can continue this process as desired.

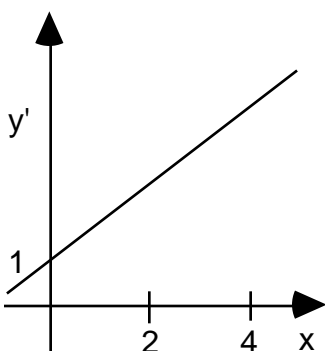


In other words, as long as the velocity function is constant or piecewise-constant, the position graph may be drawn as a linear or piecewise-linear function.

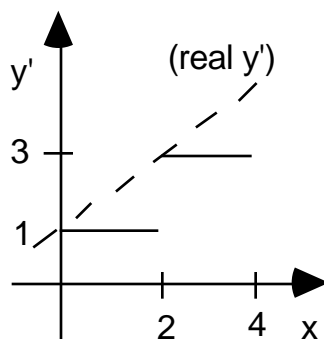
But we must return to the more general problem when the velocity or rate equation is not piecewise-constant. Of course, this is the much more natural situation. Cars do not generally travel at 30 mph and then 50 mph without traveling at 34 mph, 46 mph, and all of the other values between 30 and 50 at least momentarily. (This fact is usually obvious to most students, even though they were quite ready to deal with the artificial examples above for the sake of simplicity!) Let us first tackle an elementary equation with simple numbers. We will denote our velocity or rate equation by y' , a function of x , and we want to find the associated position function $y = f(x)$. Euler's method is based on this very simple idea: Approximate the rate equation with a piecewise-constant function.

For example, let $y' = x + 1$ for the interval where $x \in [0, 4]$. To begin, let the step size = 2.

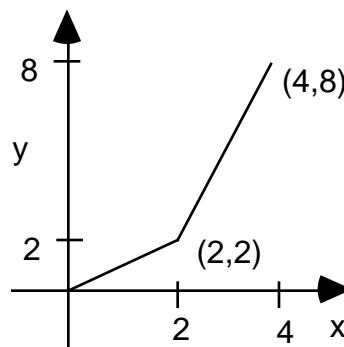
rate equation y'



approximation to y'

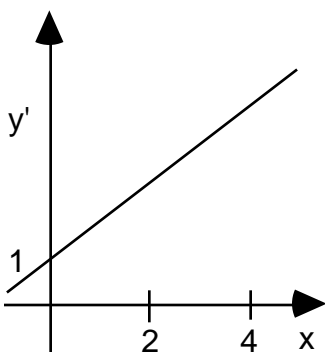


position function

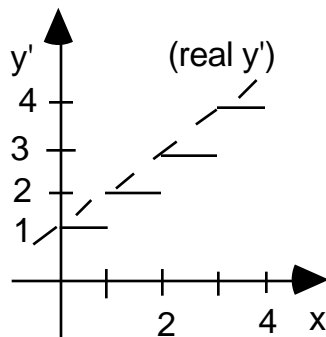


For a better approximation, choose a smaller step size, say step size = 1.

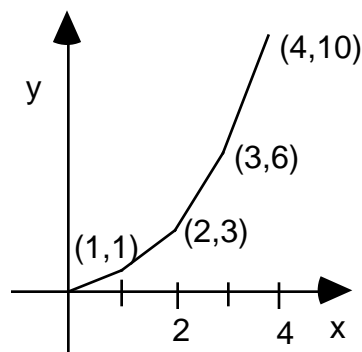
rate equation y'



approximation to y'



position function



Additionally, we can express each piecewise-defined function analytically as well. The exercise helped to emphasize the correspondence between the rate and the slope of the position function as well as reviewing the proper notation. Note that the endpoints are omitted on the rate equation, but they may be included at either endpoint of the position function.

$$y' = \begin{array}{l} 1, \quad 0 < x < 1 \\ 2, \quad 1 < x < 2 \\ 3, \quad 2 < x < 3 \\ 4, \quad 3 < x < 4 \end{array} \quad y = \begin{array}{l} x, \quad 0 \leq x \leq 1 \\ 2x - 1, \quad 1 \leq x \leq 2 \\ 3x - 3, \quad 2 \leq x \leq 3 \\ 4x - 6, \quad 3 \leq x \leq 4 \end{array}$$

It is evident that a step size of 1 gives a function y which appears to be closer to a smooth curve. In addition, note that the final value of y (when $x = 4$) is larger. This is due to the fact that the rate is continuously increasing. A better approximation of y' given by a smaller step size reflects this more accurately.

So what would the graph of y look like and what would the final value of y be if a step size of $1/2$ were chosen? You're welcome to construct this if you wish, but I choose to avoid that task. Fortunately, a programmable graphing calculator gives us the option to avoid the tedious work. In fact, it is the programming capabilities of current calculators that have allowed Euler's method to regain some popularity in some calculus courses. The following program is suitable for both the TI-81 and TI-82 calculators.

Program: EULER

0 K

Fn-Off

ClrDraw

This allows us to enter initial values of the function.

Disp "INIT X="

Input X

Disp "Y(INIT X)="

Input Z

Disp "STEP ="

Here, enter a step size, and the program calculates how many linear pieces there will be to the graph of y.

Input H

(Xmax - Xmin)/H N

Lbl 1

X A

Z B

This adds the change in y to the old value to calculate the new y.

Z + HY₁ Z

X + H X

This draws each linear piece of y, connecting the old (x,y) to the new point.

Line(A,B,X,Z)

K + 1 K

If K < N

Goto 1

This displays the terminal coordinates of the graph of y.

Pause

Disp "X="

Disp X

Disp "Y="

Disp Z

Before running the program, set Xmin and Xmax as desired and choose an appropriate range for y. X(Initial) must equal Xmin. Enter the rate equation into y₁.

To see the preceding example specifically, enter $y_1 = x + 1$, set Xmin = 0, Xmax = 4, Ymin = 0, and Ymax = 13. Execute the program. Enter X(INIT) = 0 and Y(INIT X) = 0. Choose step size = 2. The resulting graph should resemble the first graph for y shown above. Press again to see the final coordinates (4, 8). Now run the program again with step size = 1 to obtain (4, 10) for the final coordinates. Run the program several more times with the following step sizes. You should be able to create the following chart.

<u>Step Size</u>	<u>final y (at x = 4)</u>
2	8
1	10
0.5	11
0.1	11.8
0.05	11.9
0.01	11.98

This table may be extended as desired, but this already gives a great deal of information. First, it provides a nice introduction to the concept of a limit. What would you guess the real value of y should be at $x = 4$ if the actual rate equation could be used? Secondly, note what happened to the y -values as the step size decreases from 1 to 0.1 to 0.01. What would you expect to find for y if the step size were 0.001? If we assume the same pattern for step size 0.5 and 0.05, what would you expect to find for y with step size = 0.005? (Check it!)

We can also use Euler's method and the calculator program to study the general relationships between a rate equation y' and the corresponding position function y . For example, what if $y' > 0$? And if $y' < 0$? What happens to y if $y' = 0$? Can anything be said about y if y' has a maximum or a minimum point? How does y change if a different initial value of y is chosen?

Generate the graph of y for the following rate equations and try to answer the questions above:

- (1) $y' = 4 - x$ for $x \in [0, 6]$ with $y(0) = 0$
- (2) $y' = 4x - x^2$ for $x \in [0, 8]$ with $y(0) = 0$
- (3) $y' = -(x - 1)(x - 4)$ for $x \in [0, 6]$ with $y(0) = 0$
- (4) $y' = \cos x$ for $x \in [0, 6.28]$ with $y(0) = 0$
- (5) $y' = \frac{3}{x^2 + 1}$ for $x \in [0, 6]$ with $y(0) = 0$

This is only a beginning. The possibilities are endless.

As stated earlier, my calculus class spent some time at the beginning of the course studying Euler's method and using the calculator program. Students have gained a good, though elementary, understanding of a function and its rate of change. In addition, we had an introduction to limits, the importance of step sizes and approximations, increasing and decreasing functions, and maximum and minimum points – all before mentioning the word "derivative!" 🍷