MAKING CONNECTIONS USING PASCAL'S TRIANGLE, SERIES, AND MATHEMATICAL INDUCTION

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Pascal's Triangle is rich with patterns. This triangular array of numbers seems to be an inexhaustible source for making mathematical connections. In this note we will integrate Pascal's Triangle with series and use mathematical induction as a means for validating our connections. The use of mathematical induction will serve as a prototype for validating other series which can be easily designated.

We recall several attributes of Pascal's triangle. First, Pascal's Triangle can be represented by using binomial coefficients of the form $\binom{n}{r}$ where n and r are whole numbers with $n \geq r$. By definition:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Thus,

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            1
         1     1
      1     2     1
   1     3     3     1
1     4     6     4     1
            ⋮
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is the same as

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\ & & & \vdots & & & \end{array}$$

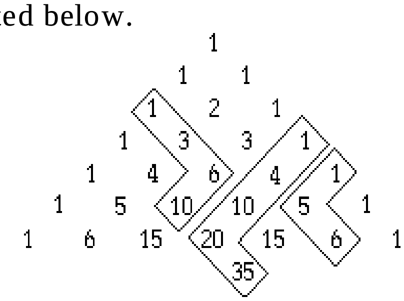
We also recall: $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

Visually, this can be seen:

$$\begin{array}{ccc} \binom{n}{r} & & \binom{n}{r+1} \\ \diagdown & + & \diagup \\ & \binom{n+1}{r+1} & \end{array}$$

where $\binom{n}{r}$ and $\binom{n}{r+1}$ are consecutive elements on the same row of Pascal's Triangle.

Students familiar with Pascal's Triangle usually have knowledge of the "Hockey Stick" property which is illustrated below.

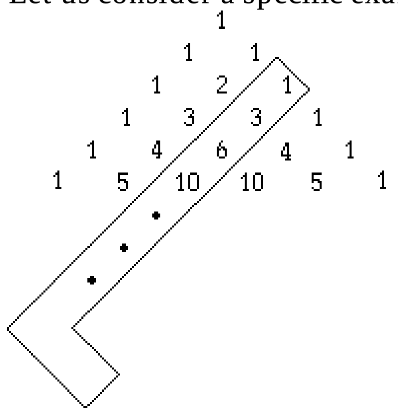


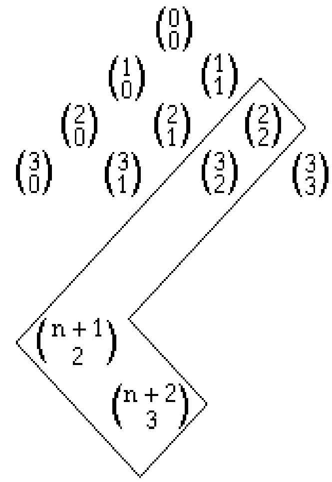
The "Hockey Sticks" drawn illustrate the following sums:

$1 + 3 + 6 = 10$
$1 + 4 + 10 + 20 = 35$
$1 + 5 = 6$

It is natural for students to ask if the "Hockey Stick" representation will always give a true result.

Let us consider a specific example:





Let P_n be the statement:

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n+1}{2}=\binom{n+2}{3}.$$

Is P_n true?

If we prove by mathematical induction we need to:

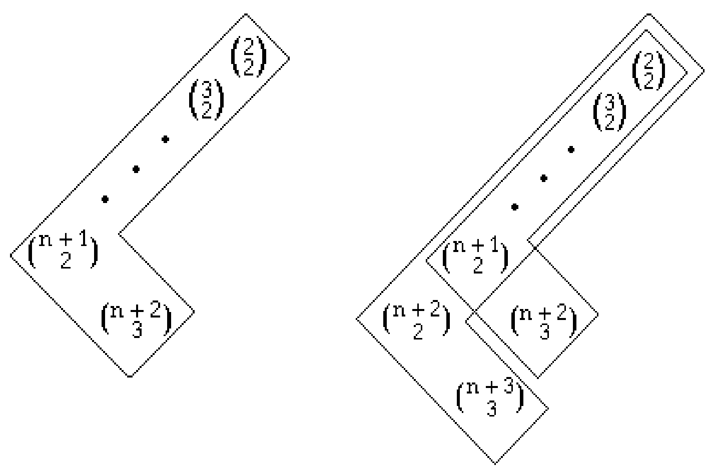
i) prove that P_1 is true.

ii) prove that if P_n is true then P_{n+1} is true.

Part (i) is easy to verify since $\binom{2}{2}=\binom{3}{3}$.

It is helpful to use a visual approach to prove that P_n implies P_{n+1}.

Assume P_n is true:



Since $\binom{n+2}{2}+\binom{n+2}{3}=\binom{n+3}{3}$ it follows that P_n implies P_{n+1}.

The same type of proof can be used to prove the generalized series of a diagonal:

$$\sum_{j=0}^{n-1}\binom{j+r}{r}=\binom{n+r}{r+1}.$$