THE ALGEBRA OF AVERAGES

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One often speaks of the *average* or *mean* of a set of numbers. These words are actually rather ambiguous when we consider that there are many types of average. Each of the following is a type of mean.

Arithmetic Mean: $\text{A.M.} = \frac{a+b}{2}$ or $\frac{a+b+c}{3}$ or $\frac{\sum_{k=1}^{n} a_k}{n}$	***Geometric Mean:*** $\text{G.M.} = \sqrt{a \cdot b}$ or $\sqrt[3]{a \cdot b \cdot c}$ or $\sqrt[n]{a_1 \cdots a_n}$
Harmonic Mean: $\text{H.M.} = \left[\frac{\frac{1}{a}+\frac{1}{b}}{2}\right]^{-1}$ or $\left[\frac{\sum_{k=1}^{n} a_k^{-1}}{n}\right]^{-1}$	***Root-Mean-Square:*** $\text{R.M.S.} = \sqrt{\frac{a^2+b^2}{2}}$ or $\sqrt{\frac{\sum_{k=1}^{n} a_k^2}{n}}$

So we might ask the question, "What do all of these have in common that makes them all **means**?" I discovered that all of the means above have a common construct. For each of them there exists a *strictly increasing* or *strictly decreasing* function, $\boldsymbol{f}$, for which the "$\boldsymbol{f}$ - mean" is determined by:

$$\text{"}\boldsymbol{f}\text{ - mean" of } (a,b) = \boldsymbol{f}^{-1}\left(\frac{\boldsymbol{f}(a)+\boldsymbol{f}(b)}{2}\right)$$

That is, the $\boldsymbol{f}$-mean of a set of numbers can be determined by applying $\boldsymbol{f}$ to each number, taking the arithmetic mean (simple average) of the resulting values, and then applying the inverse of $\boldsymbol{f}$ to the result.

For example, the Harmonic Mean of **a** and **b** is determined by the function $\boldsymbol{f}(x) = \frac{1}{x}$.

Notice first that $\boldsymbol{f}^{-1}(x) = \frac{1}{x}$

H.M. = the reciprocal of the average of the reciprocals of **a** and **b**.

$$\frac{\boldsymbol{f}(a)+\boldsymbol{f}(b)}{2} = \left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right) \text{ which simplifies to: } \frac{a+b}{2a \cdot b} \text{ and } \boldsymbol{f}^{-1}\left(\frac{a+b}{2 \cdot a \cdot b}\right) = \frac{2 \cdot a \cdot b}{a+b}$$

, the Harmonic Mean.

The Arithmetic Mean uses the identity function: $\boldsymbol{f}(x) = 1 \cdot x$ so $\text{A.M.} = \frac{1}{1}\left(\frac{1 \cdot a + 1 \cdot b}{2}\right)$

See if you can find a function, $\boldsymbol{f}$, that "determines" the Root-Mean-Square in the same way. Find such a function for the Geometric Mean. (Try to find these functions <u>before</u> you read the next page.)

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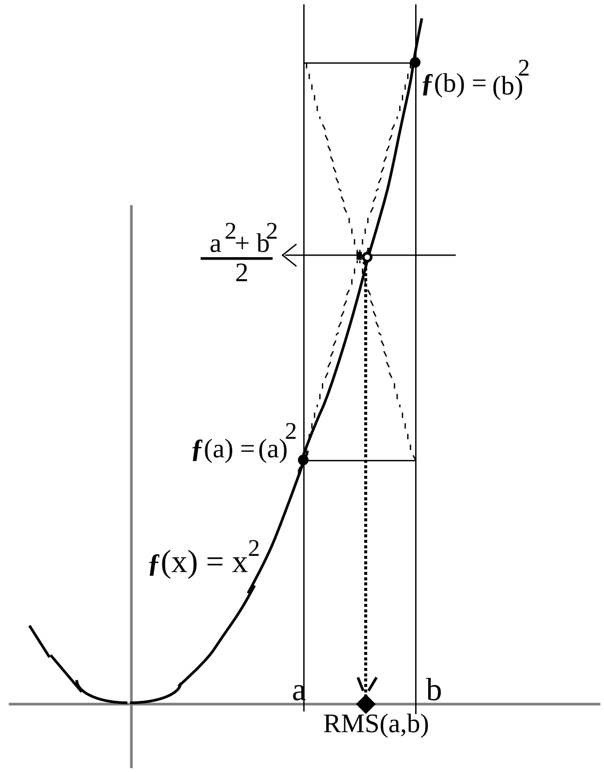
Determining means graphically from $\boldsymbol{f}$.

One can get a better feeling for the nature of means by graphing the $\boldsymbol{f}$ function and observing that the average of $\boldsymbol{f}(a)$ and $\boldsymbol{f}(b)$ crosses the curve $\boldsymbol{f}$ at a point directly above the $\boldsymbol{f}$ **-** average of **a** and **b**.

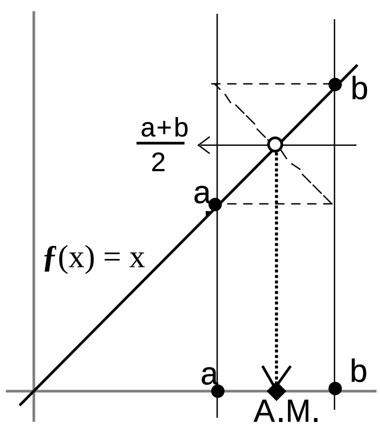


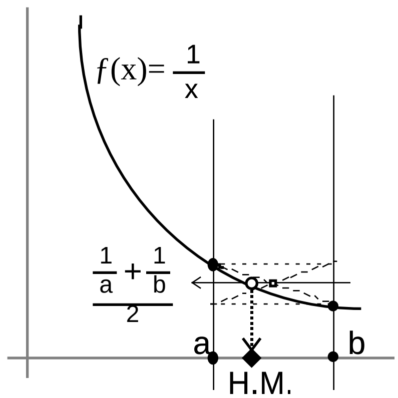
In the graph at the right, the function $\boldsymbol{f}(x) = x^2$ is sketched. If we average the heights, $\boldsymbol{f}(a)$ and $\boldsymbol{f}(b)$, for two values, **a** and **b**, ***the value of* x *for which:***

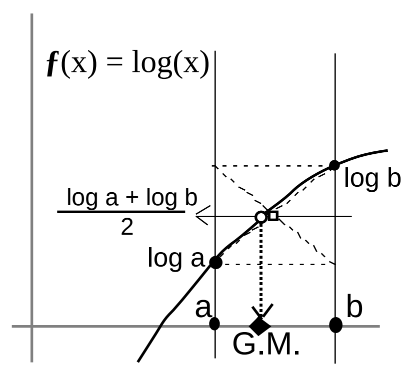
$$\boldsymbol{f}(x) = \frac{\boldsymbol{f}(a) + \boldsymbol{f}(b)}{2} = \frac{a^2 + b^2}{2}$$

is the Root Mean Square of **a** and **b**.
That is $\boldsymbol{f}(\text{RMS}(a,b))$ = y-coordinate of the intersection of the diagonals.

Observe in the figures below how the graphs of $\boldsymbol{f}(x) = x$, $\boldsymbol{f}(x) = \frac{1}{x}$, and $\boldsymbol{f}(x) = \log(x)$, can be used to derive the Arithmetic, Harmonic and Geometric means, respectively.







Notice that the geometric mean is determined by the function: $\boldsymbol{f}(p) = \log(p)$

$$\text{GM} = \text{antilog}\left(\frac{\log(a) + \log(b)}{2}\right) = \text{antilog}\left[\log(a \cdot b)^{1/2}\right] = \sqrt{a \cdot b}$$

See if you can prove each of the following formulas that relate the four means to each other and to M.D., the "Mean Difference", $\frac{b-a}{2}$.

1. H.M. $= \frac{(G.M.)^2}{A.M.}$
2. $(R.M.S.)^2 + (G.M.)^2 = 2 \cdot (A.M.)^2$
3. $GM[a,b] = GM[\,AM(a,b), HM(a,b)\,]$
4. H.M. ≤ G.M. ≤ A.M. ≤ R.M.S.
   with equality only when the numbers being averaged are identical.
5. $(A.M.)^2 - (M.D.)^2 = (G.M.)^2$
6. $(A.M.)^2 + (M.D.)^2 = (R.M.S.)^2$
7. $H.M.(a,b) = \frac{2a \cdot b}{a+b}$ and
   $H.M.(a,b,c) = \frac{3a \cdot b \cdot c}{a \cdot b + b \cdot c + c \cdot a}$

8. If $a + b\,x + c\,x^2 = 0$, $HM(r_1, r_2) = \frac{2 \cdot r_1 \cdot r_2}{r_1 + r_2} = -\frac{2 \cdot a}{b}$

   And in general:

$a + 1x = 0$	HM of roots = $-\frac{a}{1}$	GM of roots = $-a$
$a + bx + x^2 = 0$	HM of roots = $-\frac{2a}{b}$	GM of roots = $\sqrt{a}$
$a + bx + cx^2 + x^3 = 0$ (see 7.)	HM of roots = $-\frac{3a}{b}$	GM of roots = $-\sqrt[3]{a}$
$a+bx+cx^2+dx^3+x^4 = 0$	HM of roots = $-\frac{4a}{b}$	GM of roots = $\sqrt[4]{a}$
$a+bx+cx^2 + \ldots + x^n = 0$	HM of roots = $-\frac{na}{b}$	GM of roots = $(-1)^n \sqrt[n]{a}$

Write a formula for the AM and RMS of the roots of each polynomial.

9. Investigate the function: $f(x) = \frac{1}{x^2}$

   and show that its "mean" is equivalent to $\frac{(G.M.)^2}{R.M.S.}$

10. Investigate the function: $f(x) = \sqrt{x}$
    and simplify its "mean" in terms of other means.

I encourage the reader to seek other functions, f, that could be used to formulate additional means and also look for possible applications. Send your derivation and any other connections to this author at the address inside the front cover. ✍

For further information, refer to **THE GEOMETRY OF MEANS** article in the IMSA Math Journal; Volume II; Fall 1993 for connections to ways means occur in the study of geometry. ✍