THE GEOMETRY OF MEANS

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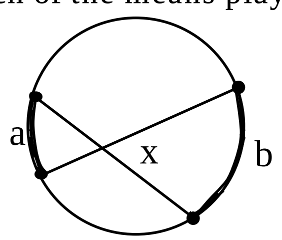
One often speaks of the *average* or *mean* of a set of numbers. These words are actually ambiguous when we consider that there are many types of average. Each of the following is a type of mean. Although means are numerical or algebraic in nature, they have applications and, in fact models, in geometry.

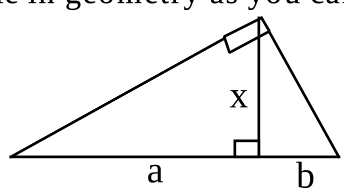
Arithmetic Mean: $\text{A.M.} = \frac{a+b}{2}$ or $\frac{a+b+c}{3}$ or $\frac{\sum_{k=1}^{n} a_k}{n}$	***Geometric Mean:*** $\text{G.M.} = \sqrt{a\cdot b}$ or $\sqrt[3]{a\cdot b\cdot c}$ or $\sqrt[n]{a_1 \cdot \cdots \cdot a_n}$
Harmonic Mean: $\text{H.M.} = \left[\frac{\frac{1}{a}+\frac{1}{b}}{2}\right]^{-1}$ or $\left[\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right]^{-1}$ or $\left[\frac{\sum_{k=1}^{n} a_k^{-1}}{n}\right]^{-1}$	***Root-Mean-Square:*** $\text{R.M.S.} = \sqrt{\frac{a^2+b^2}{2}}$ or $\sqrt{\frac{a^2+b^2+c^2}{3}}$ $= \sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}$ or $\sqrt{\frac{\sum_{k=1}^{n} a_k^2}{n}}$

Each of the means plays a role in geometry as you can see in the theorems below.



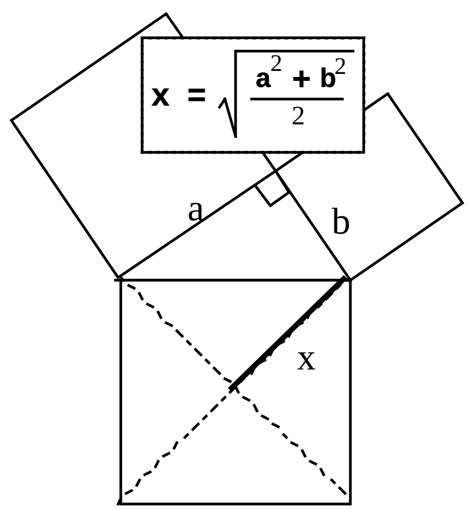
$$x = \frac{a+b}{2}$$

A chord-chord angle is



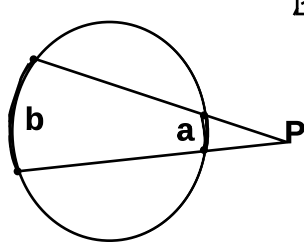
$$x = \sqrt{a\cdot b}$$

The altitude to the hypotenuse of a right triangle is the geometric mean of the parts of the hypotenuse.



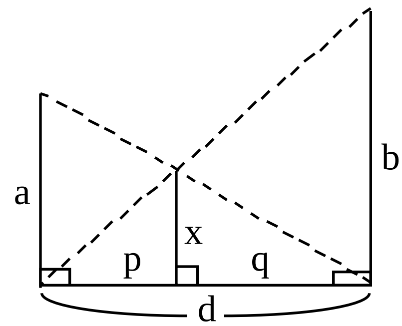
Another mean is the "Mean Difference"
It is the average of **b** and **– a.**

$$\text{M.D.}(a,b) = \frac{b-a}{2}$$



The criss-cross wire problem:

Two poles with heights **a** and **b** are **d** units apart. The top of each is connected to the bottom of the other with a wire. How far above the ground do the wires cross ?



The solution to this problem relies on the two pairs of similar triangles :
[Notice this result is independent of **d** and only requires that **a**, **b**, and **x** are from parallel segments.]

$$\frac{x}{a} = \frac{q}{p+q} \quad \text{and} \quad \frac{x}{b} = \frac{p}{p+q} \quad \text{lead to} \quad \frac{x}{a} + \frac{x}{b} = \frac{q}{p+q} + \frac{p}{p+q} = 1$$

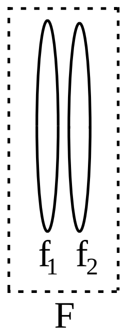
$$\text{thus:} \quad \frac{x}{a} + \frac{x}{b} = 1 \quad \text{so that} \quad \frac{1}{x} = \frac{1}{a} + \frac{1}{b} \quad \text{or} \quad x = \frac{ab}{a+b}$$

[Half of H.M.(a,b)]

This relationship also occurs in several laws of physics such as:

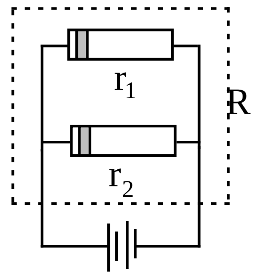
The thin lens law:
Lenses with focal length f_1 and f_2 have combined focal length F, where:

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$



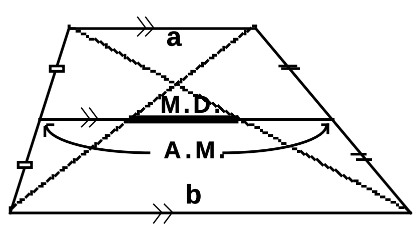
The resistance law:
A parallel resistance circuit using resistances r_1 and r_2 has a total resistance, R, where:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

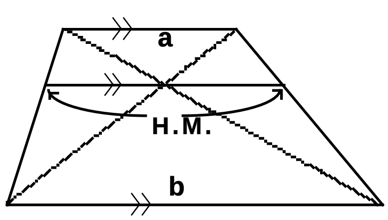


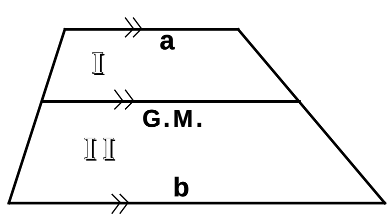
See if you can find other examples of means that exist in geometric or real life contexts.

Means in a trapezoid:

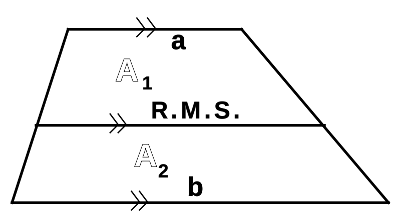


A common problem in geometry involves the median of a trapezoid.
The median of a trapezoid is the *average* of the bases.
The segment joining the midpoints of the diagonals is the *mean difference*. See if you can verify the presence of the other means in other segments parallel to the bases in the trapezoids below.





Trap I ~ Trap II

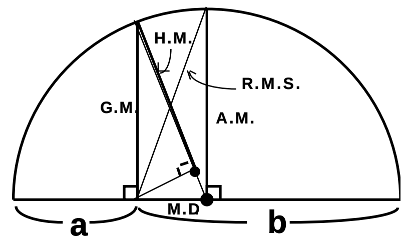


$Area_1 = Area_2$

A very important inequality relates the four means.

H.M. ≤ G.M. ≤ A.M. ≤ R.M.S.

$$\left[\frac{\frac{1}{a}+\frac{1}{b}}{2}\right]^{-1} \le \sqrt{a\cdot b} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$



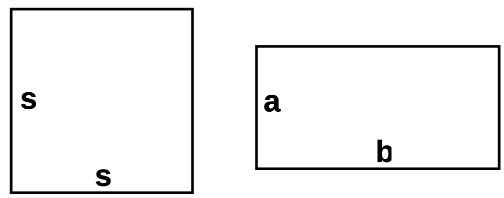
(Equality holds only when the numbers averaged are identical.)

The proof of this inequality is evident in the geometric model at the right.

PROBLEMS:

1. Given a square and a rectangle:

   See if you can verify the following:



- If the perimeters of the square and rectangle are equal, then: $s = \frac{a+b}{2}$
- If the areas of the square and rectangle are equal, then: $s = \sqrt{a\cdot b}$
- If the ratios of area to perimeter, $\left\{\frac{A_{sq}}{P_{sq}} = \frac{A_{rect}}{P_{rect}}\right\}$ are equal, then: $s = \frac{2a\cdot b}{a+b}$
- If the diagonal of the square is the rectangle then: $s = \sqrt{\frac{a^2+b^2}{2}}$

2. Similar relationships can be found for the dimensions, **s**, of a cube and **a**, **b**, and **c** of a rectangular box. See if you can find and verify each of these properties for solids where **s** is a mean of **a**, **b**, and **c**.

3. Given two adjacent squares. Find the distance between their centers and point out any means that you encounter along the way. ✍