FINDING MOMENTS OF INERTIA, *DISCRETELY*

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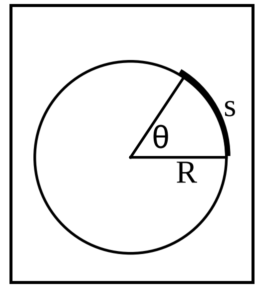


Consider an object traveling in a circular path of radius, **R**. If it rotates through an angle θ, its displacement, **s**, is related to the angle, θ, by **s = R**θ. If we look at the object's velocity, **v**, as ds / dt then **v = R dθ/dt**. We call **dθ/dt**, ω, angular velocity, measured in radians / s ($\omega = 2\pi R/T$ or $2\pi Rf$). If we look at the object's acceleration, **a = dv/dt = R dω/dt**. We call **dω/dt**, α, angular acceleration, measured in radians / s^2. Thus we have:

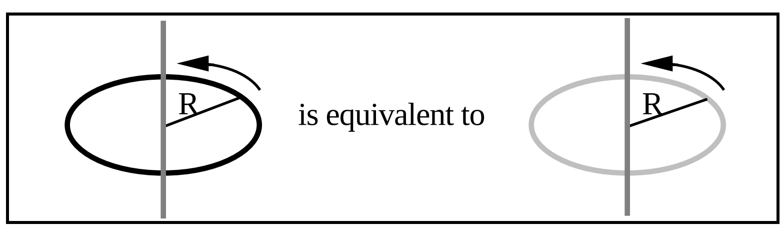
$$s = R\theta$$
$$v = R\omega$$
$$a = R\alpha$$

Suppose this object of mass **m** is going around in a circle and is being accelerated by a force, **F**, which acts to **R**. Newton's 2nd Law says the body will accelerate such that **a = F/m**. Let's write that as **F = ma**. If we multiply both sides by **R**, we get **RF = mRa**. If we convert **a** to rotational notation, **RF = mRR**α or $mR^2\alpha$.

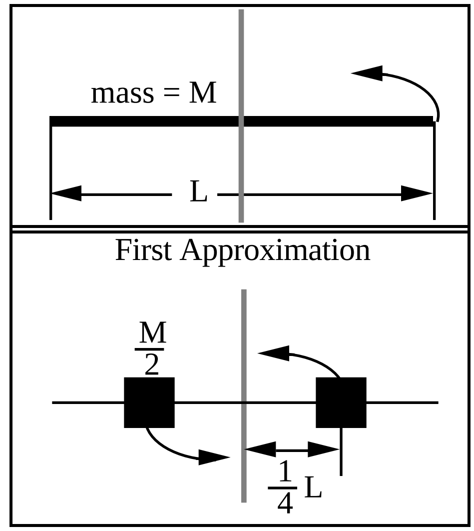
We call **RxF**, *torque* and give it the symbol τ. MR^2 is called the object's rotational inertia, often referred to as the "moment of inertia" ($I = mR^2$). Newton's 2nd Law for rotating bodies is therefore written τ = **I**α

The moment of inertia of a body is easy to calculate if <u>all</u> of the body is at the same distance, **R**, from the point of rotation. A thin ring, for example, can be considered to be the sum of a large number of evenly spaced point objects, each the same distance from the point of rotation. Its total inertia, **I**, would just be the sum of the individual mass' ΔI. If we consider the ring as consisting of **n** equal mass pieces (each **M/n**), then each mass would have a $\Delta I = (M/n)\, R^2$. The total inertia, **I**, would be the sum of all the individual moments:

$$I = \sum \frac{M}{n} R^2 \quad = M R^2$$



But, suppose all of the mass of a spinning object was not at the same distance from the center of rotation. Consider a rod of length, **L**, and mass, **M**, rotated about its midpoint.



As a first approximation, let’s divide the rod into 2 pieces, each with mass $\frac{\mathbf{M}}{\mathbf{2}}$, located at $\frac{\mathbf{1}}{\mathbf{4}}$ **L** from the midpoint. We’ll examine the left half of the rod and double the results to include the right half.

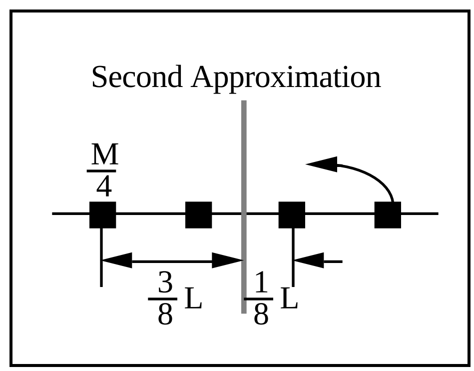
$$\Delta \mathbf{I_1} = \frac{\mathbf{M}}{\mathbf{2}}\left(\frac{\mathbf{L}}{\mathbf{4}}\right)^{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{32}}\ \mathbf{ML^2}$$

and the total moment of inertia for both pieces is

$$\mathbf{I = 2}\ \Delta \mathbf{I_1} = \frac{\mathbf{1}}{\mathbf{16}}\ \mathbf{ML^2}$$

Not too bad; the accepted value is $\frac{1}{12}$ ML^2. Let's try a better approximation.

Divide the rod into **4** pieces, each $\frac{\mathbf{M}}{\mathbf{4}}$, and locate two of the pieces at $\frac{\mathbf{1}}{\mathbf{8}}$ **L** and the other two at $\frac{\mathbf{3}}{\mathbf{8}}$ **L** as shown.



$\Delta \mathbf{I_1} = \frac{\mathbf{M}}{\mathbf{4}}\left(\frac{\mathbf{L}}{\mathbf{8}}\right)^{\mathbf{2}}$ for one of the inner pieces.

$\Delta \mathbf{I_2} = \frac{\mathbf{M}}{\mathbf{4}}\left(\frac{\mathbf{3\,L}}{\mathbf{8}}\right)^{\mathbf{2}}$ for one of the outer pieces.

$$\mathbf{I = 2}\ (\Delta \mathbf{I_1} + \Delta \mathbf{I_2}) = \mathbf{2}\frac{\mathbf{M}}{\mathbf{4}}\left[\left(\frac{\mathbf{L}}{\mathbf{8}}\right)^{\mathbf{2}} + \left(\frac{\mathbf{3\,L}}{\mathbf{8}}\right)^{\mathbf{2}}\right] = 2\frac{M}{4}\left(\frac{L}{8}\right)^2\left[1^2 + 3^2\right] = \frac{5}{64}\,\mathbf{M\,L^2}$$

With the rod divided into 8 pieces (located at $\frac{\mathbf{1}}{\mathbf{16}}$ **L,** $\frac{\mathbf{3}}{\mathbf{16}}$ **L,** $\frac{\mathbf{5}}{\mathbf{16}}$ **L,** and $\frac{\mathbf{7}}{\mathbf{16}}$ **L** on either side), the total moment of inertia becomes

$$\mathbf{I = 2}\ \frac{M}{8}\left(\frac{L}{16}\right)^2\left[1^2 + 3^2 + 5^2 + 7^2\right]$$

In general, if we divide the rod into n pieces (n is always even to preserve the symmetry),

$$\mathbf{I = 2}\ \frac{M}{n}\left(\frac{L}{2n}\right)^2\left[1^2+3^2+5^2+\ldots+(n-1)^2\right]$$

But what is the sum of $1^2 + 3^2 + 5^2 + \ldots + (n-1)^2$? Let's examine this series more closely.

Series Sum	1	10	35	84	165	286
n	2	4	6	8	10	12

If we look at the first order differences in the series sum we get

		9		25		49		81		121	
Series Sum	1		10		35		84		165		286
n	2		4		6		8		10		12

If we look at the second order differences,

			16		24		32		40		
		9		25		49		81		121	
Series Sum	1		10		35		84		165		286
n	2		4		6		8		10		12

But when we look at the third order differences,

				8		8		8			
			16		24		32		40		
		9		25		49		81		121	
Series Sum	1		10		35		84		165		286
n	2		4		6		8		10		12

we see that they're the same value. This tells us that the relationship between the series sum and n is cubic (third order) and has the form: $f(n) = An^3 + Bn^2 + Cn + D$. Let's get four independent equations from the data table above and use them to discover the values of A, B, C, and D.

$$\left.\begin{array}{ll}(1) & 1 = 8A + 4B + 2C + D\\ (2) & 10 = 64A + 16B + 4C + D\\ (3) & 35 = 216A + 36B + 6C + D\\ (4) & 84 = 512A + 64B + 8C + D\end{array}\right\}$$

Subtracting Eq.

$$\begin{array}{lll} (2) - (1) & (5) & \left\{ \begin{array}{l} 9 = 56A + 12B + 2C \\ 25 = 152A + 20B + 2C \\ 49 = 296A + 28B + 2C \end{array} \right. \\ (3) - (2) & (6) & \\ (4) - (3) & (7) & \\ (6) - (5) & (8) & \left\{ \begin{array}{l} 16 = 96A + 8B \\ 24 = 144A + 8B \end{array} \right. \\ (7) - (6) & (9) & \\ (9) - (8) & (10) & 8 = 48A \qquad A = \frac{1}{6} \end{array}$$

subst in (8) $\quad 16 = 16 + 8B \qquad B = 0$

subst in (5) $\quad 9 = 9\frac{1}{3} + 2C \qquad C = -\frac{1}{6}$

subst in (1) $\quad 1 = \frac{8}{6} - \frac{2}{6} + D \qquad D = 0$

So, for our series sum, $f(n) = \frac{n^3}{6} - \frac{n}{6}$ or $\frac{n(n^2-1)}{6}$

Now, let's return to our moment of inertia problem. If we divide our rod into n pieces and distribute them appropriately, the sum of all of their individual moments of inertia is

$$\mathbf{I = 2}\ \frac{M}{n}\left(\frac{L}{2n}\right)^2\left[1^2 + 3^2 + 5^2 + \ldots + (n-1)^2\right]$$

which we now see as being

$$\mathbf{I = 2}\ \frac{M}{n}\left(\frac{L}{2n}\right)^2\left[\frac{n(n^2-1)}{6}\right]$$

As $n \rightarrow \infty$, $\frac{n(n^2-1)}{6} \rightarrow \frac{n^3}{6}$, and $\mathbf{I} = \frac{\mathbf{1}}{\mathbf{12}}\ \mathbf{M\,L^2}$. ✍